Homotopy pullbacks for n-categories

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Abstract

We present a notion of homotopy pullback in the sesqui-category n-Cat of strict n-categories, strict n-functors and weak n-transformations; moreover we show that such a construction satisfies also a 2-dimensional universal property with respect to n-modifications. This lead us to introduce the new notion of sesqui^2-category.

1 Introduction

Amongst the different approaches used in dealing with higher dimensional categorical structures, the inductive-enriched is probably one of the less developed. This fact is not surprising at all, as there are indeed very good reasons that lead the majority to adopt other viewpoints. In fact, such an approach is not really suitable for making calculations easy, as the inductive process has to be unroll in order to get explicit. Neither it fits with Internal Category Theory, which would make it useful in applications. Finally there are other approaches, as the simplicial one, that can take advantage of an already well developed theory.
Nevertheless it seems that there are also good reasons to take the inductive-enriched approach.
There are issues indeed that could be more treatable under this perspective: for instance some of the coherence issues detailed in this paper, as axioms for lax n-transformations, are dealt with inductively in a natural and simple way.
More deeply, in certain situations the inductive-enriched point of view seems to be closely related with how and why certain properties and structures are defined. (see [KMV08a] for an application).

This paper is about homotopy pullbacks for strict-n-categories. We are interested in such a construction in order to get a notion of h-kernel, and hence a notion of exactness to be used in developing homotopical (and homological) algebra for pointed strict n-groupoids[1] [KMV08b]. Further developing will involve also the study of homotopy colimits (see [DF04]), in order to establish a connection with homotopy-theoretical issues.
The idea of our definition recaptures homotopical aspects from the topological standard h-pullback [Mat76a]. In fact, having in mind the geometric realization

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[1] See [Gra94] for a detailed account on homotopy theories in a sesquicategorical setting.

functor for (eventually pointed n-)groupoids, one can tentatively emulate the constructions involving points, paths, homotopies of paths etc. of a space with the objects, the arrows and the higher dimensional cells of a n-groupoid. Hence, let us suppose we are given two maps $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{C} \to \mathcal{B}$ in our favorite topological category (say Top, Top_*, $\mathrm{CW}-\mathrm{cplx}$ etc.) their h-pullback is the space

$$\mathcal{P} = \{(a, \theta, c) \in \mathcal{A} \times \mathcal{B}^I \times \mathcal{C} : F(a) = \theta(0), G(c) = \theta(1)\}$$

where $\mathcal{B}^I$ is taken with the compact-open topology and $\mathcal{P}$ is topologized as a subset of $\mathcal{A} \times \mathcal{B}^I \times \mathcal{C}$.

The corresponding construction for two n-functors $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{C} \to \mathbb{B}$ is the h-pullback $\mathbb{P}$, and has objects the triples (a, t, c) where a and c are objects of $\mathbb{A}$ and $\mathbb{C}$ respectively, while $t : F(a) \to G(c)$ is a 1-cell of $\mathbb{B}$. Similarly one can define explicitly the 1-cells of $\mathbb{P}$, namely triples such as

$$(f : a \to a', \epsilon : F(f) \circ t' \Rightarrow t \circ G(g), g : c \to c').$$

It is clear that, as the dimension of the cells raises up, the complexity of their description increases. Inductive approach permits to deal with this situation quite easily, since it is possible to show that the homs of a h-pullback of n-categories are themselves h-pullbacks of dimension $n-1$.

The paper is organized as follows. Next section recalls the inductive definition of strict n-categories, introduces a notion of lax-n-transformation that gives the category nCat a structure of a sesqui-category ([Str96, MF08]), finally the standard h-pullback of n-categories is constructed. The following section is devoted to developing three-dimensional aspects of nCat, this is done by introducing the notion of lax-n-modification thus giving nCat a structure of a sesqui^2-category; the end of the section describes a 2-dimensional universal property that defines of h^2-pullbacks in nCat; *Theorem 3.13* show that h-pullbacks of n-categories satisfy the 2-dimensional universal property referred above. In other words n-Cat is a sesqui^2-category with h-pullbacks. The last section analyzes the different environment encountered throughout the paper: the notion of sesqui-category is recalled while the new notion of sesqui^2-category is introduced and characterized.

The present work is based on the PhD Thesis of the author [Met08b], under the supervision of S. Kasangian and E. M. Vitale.

2 Strict n-categories

In the first part we begin by recalling a standard construction of the category n-Cat, of (small) strict–n-categories and their morphisms, inductively enriched over the category $(n-1)$-Cat (see for instance [Str87]). Then we give n-Cat a structure of *sesqui-category*, in order to take into account the 2-morphisms, namely lax–n-transformations, and their compositions. Indeed this is the minimal setting in which to embed our notion of exact sequence. Nevertheless in developing the theory we are forced to consider a slightly richer structure that we have called *sesqui*^{2}*–category*, that extends that of *sesqui-category* by considering 3-morphism namely lax–n-modifications, together with some kind of compositions. For the basics on sesqui-categories and sesqui^2–categories we refer to the last section.

2.1 The category n-Cat

For $n = 0, 1$ we can safely consider the usual category of small sets and categories respectively. Hence let us suppose $n > 1$.
A (strict) n-category $\mathbb{C}$ consists of a set of objects $\mathbb{C}_0$, and for every pair $c_0, c_0' \in \mathbb{C}_0$, a ($n$-1)-category $\mathbb{C}_1(c_0, c_0')$. Structure is given by morphisms of (n-1)-categories:

$$\mathbb{I} \xrightarrow{u^0(c_0)} \mathbb{C}_1(c_0, c_0) \; , \qquad \mathbb{C}_1(c_0, c_0') \times \mathbb{C}_1(c_0', c_0'') \xrightarrow{\circ^0_{c_0, c_0', c_0''}} \mathbb{C}_1(c_0, c_0'') \; ,$$

called resp. 0-units and 0-compositions, with c_0, c_0', c_0'' any triple of objects $\mathbb{C}_0$. Axioms are the usual for strict unit and strict associativity.
Notation: Cell dimension will be often recalled as subscript, as c_k is a k-cell in the n-category $\mathbb{C}$. Moreover, if

$$c_k : c_{k-1} \to c'_{k-1} : c_{k-2} \to c'_{k-2} : \cdots \to \cdots : c_1 \to c_1' : c_0 \to c_0',$$

c_k can be considered as an object of the $(n-k)$-category

$$\Big[\cdots \big[[\mathbb{C}_1(c_0, c_0')]_1(c_1, c_1') \big]_1 \cdots \Big]_1 (c_{k-1}, c'_{k-1}).$$

In order to avoid this quite uncomfortable notation, the latter will be renamed more simply $\mathbb{C}_k(c_{k-1}, c'_{k-1})$, while with $\mathbb{C}_k$ we will mean the disjoint union of all such. Finally, 0-subscript of the underlying set of an n-category, will be often omitted.
A morphism of n-categories is a (strict) n-functor $\mathbb{F} : \mathbb{C} \to \mathbb{D}$. It consists of set-theoretical map $F_0 : \mathbb{C}_0 \to \mathbb{D}_0$ together with morphisms of (n-1)-categories

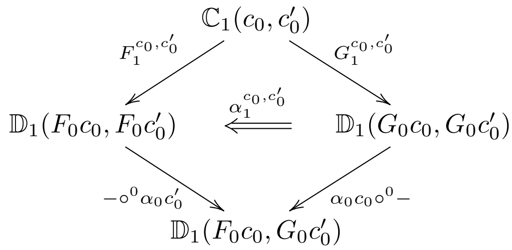
$$F_1^{c_0, c_0'} : \mathbb{C}_1(c_0, c_0') \longrightarrow \mathbb{D}_1(F_0 c_0, F_0 c_0')$$

for any pair of objects c_0, c_0' of $\mathbb{C}_0$, such that usual (strict) functoriality axioms are satisfied. Let us notice that subscripts and superscripts will be often omitted, when this does not cause confusion.
Routine calculations shows that these data organizes in a category, with finite products and terminal object defined in the obvious way [Met08b].

2.2 The sesqui–categorical structure of n-Cat

The category Set can be easily endowed with a trivial sesqui–categorical structure. For $n = 1$, the category Cat is a 2-category, with natural transformations as 2-cells. Hence it has an underlying sesqui-category.
Again we can suppose $n > 1$. For given n-functors $F, G : \mathbb{C} \to \mathbb{D}$, a lax n-transformation $\alpha : F \to G$ consists of a pair (α_0, α_1) where the first is a map $\alpha_0 : \mathbb{C}_0 \to \mathbb{D}_1$ such that $\alpha_0(c_0) = \alpha_{c_0} : Fc_0 \to Gc_0$, and $\alpha_1 = \{\alpha_1^{c_0, c_0'}\}_{c_0, c_0' \in \mathbb{C}_0}$ is

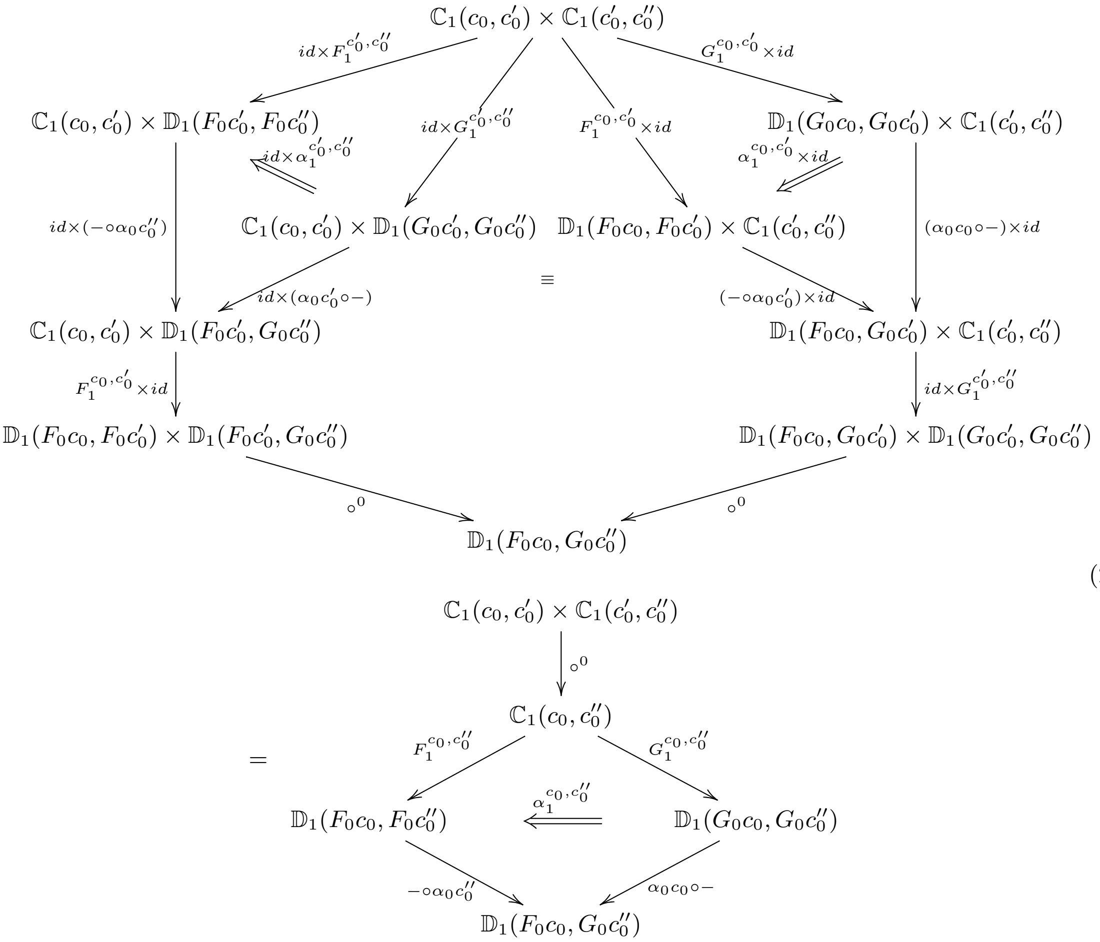
a collection of 2-morphisms of (n-1)-categories



(1)

satisfying axioms below:

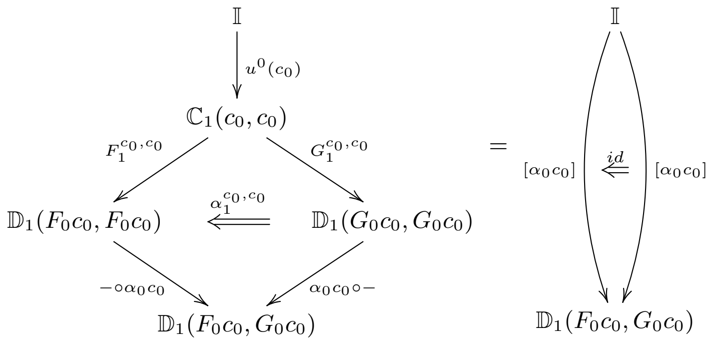
- (*functoriality w.r.t. composition*) for every triple of objects c_0, c_0', c_0'' of $\mathbb{C}_0$,



(2)

- (*functoriality w.r.t. units*) for every object c_0 of

$\mathbb{C}_0$,

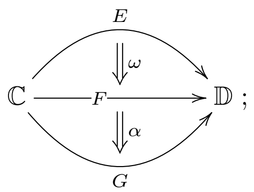


(3)

In the sequel we will refer to diagrams like (1) as to *naturality diagrams* for the 2-morphism α. A n-transformation is called strict when all $\alpha_1^{-,-}$ are identities.

n**Cat: the hom-categories**

In this section we describe, hom-categories n-Cat$(\mathbb{C}, \mathbb{D})$, once n-categories $\mathbb{C}$ and $\mathbb{D}$ are fixed.
Given the diagram:



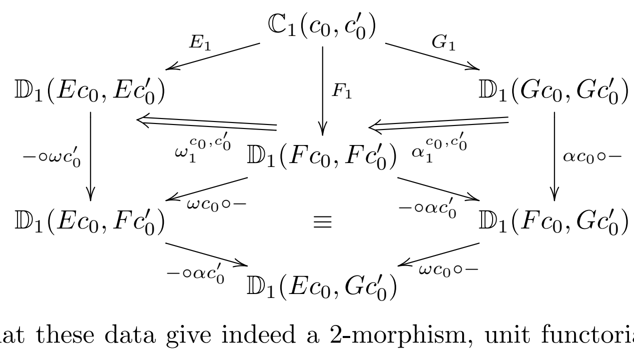
one defines a (vertical, or 1-)composition $\omega \bullet^1 \alpha : E \Longrightarrow G$ in the following way:

- for every object c_0 in $\mathbb{C}$,

$$[\omega \bullet^1 \alpha]_0(c_0) = \omega_0 c_0 \circ^0 \alpha_0 c_0 : \ Ec_0 \longrightarrow Gc_0$$

- for every pair of objects c_0, c_0' in $\mathbb{C}$, the diagram below describes

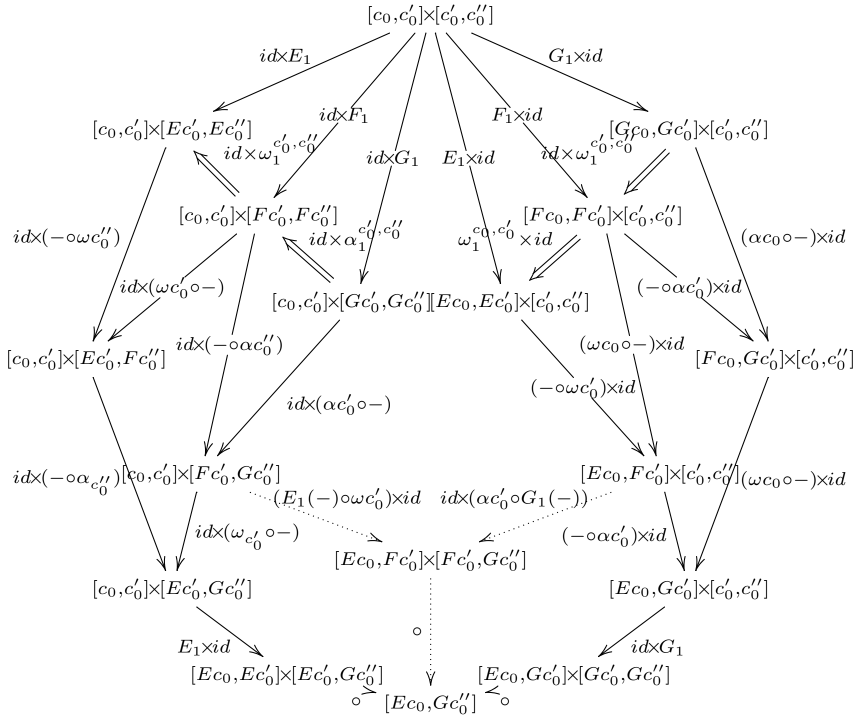
$$[\omega \bullet^1 \alpha]_1^{c_0,c_0'} = \left(\omega c_0 \circ \alpha_1^{c_0,c_0'}\right) \bullet^1 \left(\omega_1^{c_0,c_0'} \circ \alpha c_0'\right)$$



To prove that these data give indeed a 2-morphism, unit functoriality (3) and composition functoriality (2) equations must hold To this end, chose an object c_0 of $\mathbb{C}$, then

$$u(c_0)((\alpha_1^{c_0,c_0}(\omega_{c_0}\circ -))\bullet^1(\omega_1^{c_0,c_0}(-\circ\alpha_{c_0}))) \overset{(i)}{=} id_{\alpha_{c_0}}(\omega_{c_0}\circ -)\bullet^1 id_{\omega_{c_0}}(-\circ\alpha_{c_0}) \overset{(ii)}{=} id_{[\omega\alpha]_{c_0}}$$

where (i) follows for units functoriality of ω and α, while (ii) from functoriality of constant functors. This proves unit functoriality.
Concerning composition functoriality, take three objects c_0, c_0' and c_0'' in $\mathbb{C}$, and consider the following diagram:



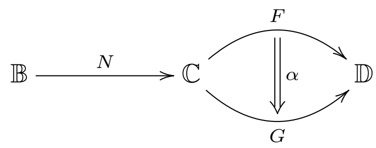
After applying the product interchange (see section 4.2) to 2-morphisms $\omega_1^{c_0,c_0'}$ and $\alpha_1^{c_0',c_0''}$, by functoriality of 2-morphisms in $(n-1)$-Cat the two sides of the diagram give $[\circ_{c_0,c_0',c_0''}](\alpha_1^{c_0,c_0''}(\omega_{c_0}\circ -)\bullet^1 \omega_1^{c_0,c-0''}(-\circ\alpha_{c_0''}))$ that is exactly $[\omega\bullet^1\alpha]_1^{c_0,c_0''}$, and this concludes the proof.
Given a morphism of n-categories $F:\mathbb{C}\xrightarrow{F}\mathbb{D}$, it is possible to define the *unit 2-cell of* F, This is denoted id_F, with $[id_F]_0(c_0)=id_{Fc_0}$ and $[id_F]_1^{c_0,c_0'}=id_{F_1^{c_0,c_0'}}$
It is straightforward to see that these give a 2-morphism, and prove the following

Proposition 2.1. *Let us fix n-categories* $\mathbb{C}$ *and* $\mathbb{D}$*. Morphisms between them and 2-morphisms between those form a category, with composition and units given above.*

nCat: the sesqui-categorical structure

In the next sections we will introduce reduced left/right compositions of morphisms and 2-morphisms of n-categories, in order to show that nCat has a canonical sesqui-categorical structure. Notice that 0Cat=Set has a trivial sesqui-categorical structure (all 2-cells are identities), while 1Cat=Cat has a canonical 2-categorical structure, that inherits a sesqui-categorical structure, forgetting horizontal composition of 2-cells. Hence we may well suppose $n>1$.

Given the situation



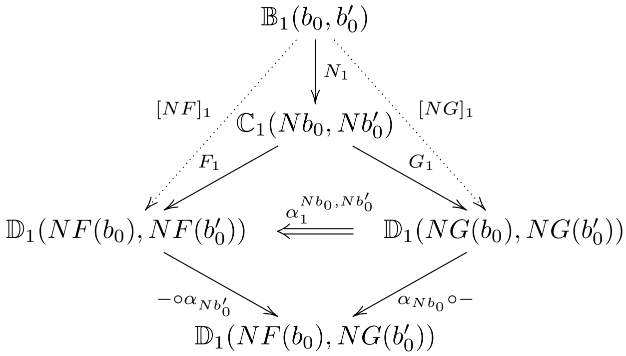
one defines reduced horizontal composition $N \bullet^0 \alpha : NF \Rightarrow NG : \mathbb{B} \to \mathbb{D}$ (or 0-composition) in the following way:

• for every object b_0 in $\mathbb{B}$,

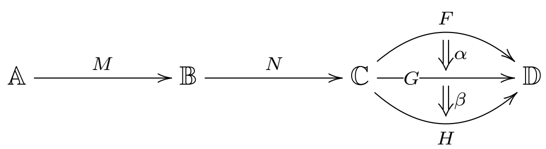
$$[N \bullet^0 \alpha]_0 = \alpha_0(N(b_0)) : F(N(b_0)) \to G(N(b_0))$$

• for every pair of objects b_0, b_0' of $\mathbb{B}$, the diagram below describes $[N \bullet^0 \alpha]_1^{b_0,b_0'}$ by means of reduced left composition in $(n-1)$Cat:

$$[N \bullet^0 \alpha]_1^{b_0,b_0'} = N_1^{b_0,b_0'} \bullet^0 \alpha_1^{Nb_0,Nb_0'}$$



To prove that these data give indeed a 2-morphism, one shows that unit and composition axioms (3) (2) (see [Met08b]).
Moreover, given the situation



in nCat, left-composition defined above satisfies axioms (L1) to (L4) of *Proposition 4.2.*

(L1)

$$Id_{\mathbb{C}} \bullet^0 \alpha = \alpha$$

Proof. Let objects c_0, c_0' of $\mathbb{C}$ be given. It is clear that

$$[Id_{\mathbb{C}} \bullet^0 \alpha]_{c_0} \overset{(def)}{=} \alpha_{Id_{\mathbb{C}}(c_0)} = \alpha_{c_0}$$

and also that

$$[Id_{\mathbb{C}} \bullet^0 \alpha]_1^{c_0,c_0'} \overset{(def)}{=} [Id_{\mathbb{C}}]_1^{c_0,c_0'} \bullet^0 \alpha_1^{c_0,c_0'} \overset{(1)}{=} Id_{\mathbb{C}_1(c_0,c_0')} \bullet^0 \alpha_1^{c_0,c_0'} \overset{(2)}{=} \alpha_1^{c_0,c_0'}$$

where (1) comes from the definition of *identity functors*, and (2) is axiom (L1) for $(n-1)$-Cat.

□

(L2)

$$MN \bullet^0 \alpha = M \bullet^0 (N \bullet^0 \alpha)$$

Proof. Let objects a_0, a_0' of $\mathbb{A}$ be given. Then

$$[MN \bullet^0 \alpha]_{a_0} \overset{(def)}{=} \alpha_{MN(a_0)} = \alpha_{N(Ma_0)} \overset{(def)}{=} [N \bullet^0 \alpha]_{Ma_0} \overset{(def)}{=} [M \bullet^0 (N \bullet^0 \alpha)]_{a_0}$$

Furthermore,

$$\begin{aligned} [MN \bullet^0 \alpha]_1^{a_0,a_0'} &\overset{(def)}{=} [MN]_1^{a_0,a_0'} \bullet^0 \alpha_1^{MN(a_0),MN(a_0')} \\ &= M_1^{a_0,a_0'} N_1^{Ma_0,Ma_0'} \bullet^0 \alpha_1^{MN(a_0),MN(a_0')} \\ &\overset{(1)}{=} M_1^{a_0,a_0'} \bullet^0 (N_1^{Ma_0,Ma_0'} \bullet^0 \alpha_1^{N(Ma_0),N(Ma_0')}) \\ &\overset{(def)}{=} M_1^{a_0,a_0'} \bullet^0 [N \bullet^0 \alpha]_1^{Ma_0,Ma_0'} \\ &\overset{(def)}{=} [M \bullet^0 (N \bullet^0 \alpha)]_1^{a_0,a_0'} \end{aligned}$$

where (1) is axiom (L2) for $(n-1)$-Cat.

□

(L3)

$$N \bullet^0 id_F = id_{NF}$$

Proof. Let objects b_0, b_0' of $\mathbb{B}$ be given. Trivially,

$$[N \bullet^0 id_F]_{b_0} \overset{(def)}{=} [id_F]_{Nb_0} = [id_{NF}]_{b_0}$$

and

$$\begin{aligned} [N \bullet^0 id_F]_1^{b_0,b_0'} &\overset{(def)}{=} N_1^{b_0,b_0'} \bullet^0 [id_F]_1^{Nb_0,Nb_0'} \overset{(1)}{=} N_1^{b_0,b_0'} \bullet^0 id_{F_1^{Nb_0,Nb_0'}} = \\ &\overset{(2)}{=} id_{N_1^{b_0,b_0'} F_1^{Nb_0,Nb_0'}} = id_{[NF]_1^{b_0,b_0'}} \overset{(def)}{=} [id_{NF}]_1^{b_0,b_0'} \end{aligned}$$

where (1) comes from the definition of identity transformation and (2) is axiom (L3) in $(n-1)$-Cat.

□

(L4)

$$N \bullet^0 (\alpha \bullet^1 \beta) = (N \bullet^0 \alpha) \bullet^1 (N \bullet^0 \beta)$$

Proof. Let objects b_0, b_0' of $\mathbb{B}$ be given. On objects:

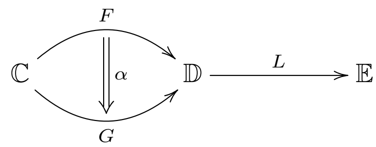
$$[N\bullet^0(\alpha\bullet^1\beta)]_{b_0} \overset{(def)}{=} [\alpha\bullet^1\beta]_{Nb_0} = \alpha_{Nb_0}\circ\beta_{Nb_0} \overset{(def)}{=} [N\bullet^0\alpha]_{b_0}\circ[N\bullet^0\beta]_{b_0} = [(N\bullet^0\alpha)\bullet^1(N\bullet^0\beta)]_{b_0}$$

On homs:

$$
\begin{aligned}
[N \bullet^0 (\alpha \bullet^1 \beta)]_1^{b_0,b_0'} &\overset{(def)}{=} N_1^{b_0,b_0'} \bullet^0 [\alpha \bullet^1 \beta]_1^{Nb_0,Nb_0'} \\
&\overset{(1)}{=} N_1^{b_0,b_0'} \bullet^0 \Big(\big(\beta_1^{Nb_0,Nb_0'} \bullet^0 (\alpha_{Nb_0} \circ -)\big) \bullet^1 \big(\alpha_1^{Nb_0,Nb_0'} \bullet^0 (- \circ \beta_1^{Nb_0,Nb_0'})\big)\Big) \\
&\overset{(2)}{=} \big(N_1^{b_0,b_0'} \bullet^0 \beta_1^{Nb_0,Nb_0'} \bullet^0 (\alpha_{Nb_0} \circ -)\big) \bullet^1 \big(N_1^{b_0,b_0'} \bullet^0 \alpha_1^{Nb_0,Nb_0'} \bullet^0 (- \circ \beta_1^{Nb_0,Nb_0'})\big) \\
&\overset{(def)}{=} \big([N \bullet^0 \beta]_1^{b_0,b_0'} \bullet^0 ([N \circ \alpha]_{b_0} \circ -)\big) \bullet^1 \big([N \bullet^0 \alpha]_1^{b_0,b_0'} \bullet^0 (- \circ [N \bullet^0 \beta]_{b_0'})\big) \\
&\overset{(3)}{=} [(N \bullet^0 \alpha) \bullet^1 (N \bullet^0 \beta)]_1^{b_0,b_0'}
\end{aligned}
$$

where (1) and (3) hold by definition of vertical composites of 2-morphisms, (2) by axiom (L4) in $(n-1)$-Cat. □

Given the situation



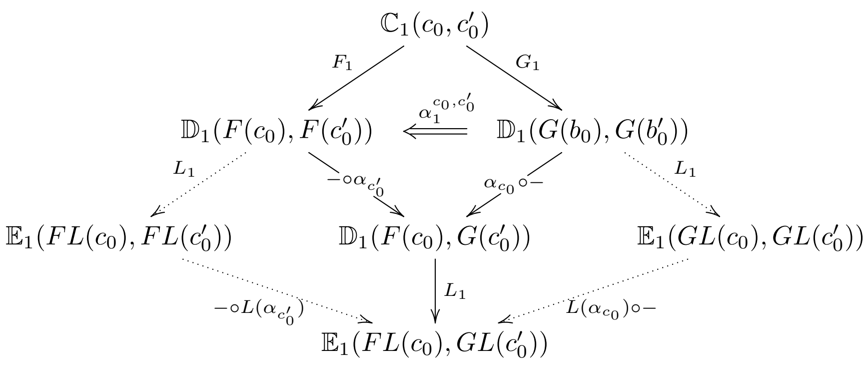
one defines reduced horizontal composition $\alpha \bullet^0 L : FL \Rightarrow GL : \mathbb{C} \to \mathbb{E}$ (or 0-composition) in the following way:

• for every object c_0 in $\mathbb{C}$,

$$[\alpha \bullet^0 L]_0 = L(\alpha_0(c_0)) : L(F(c_0)) \to L(G(c_0))$$

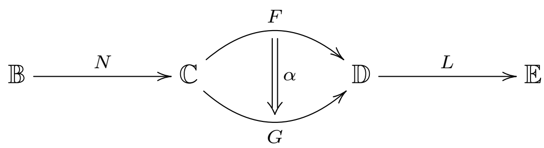
• for every pair of objects c_0, c_0' of $\mathbb{B}$, the diagram below describes $[\alpha \bullet^0 L]_1^{c_0,c_0'}$ by means of reduced right composition in $(n-1)$-Cat:

$$[\alpha \bullet^0 L]_1^{c_0,c_0'} = \alpha_1^{c_0,c_0'} \bullet^0 L_1^{Fc_0,Gc_0'}$$



Again one must show that these data give indeed a 2-morphism, i.e. that unit and composition axioms (3) (2) hold, and that right-composition defined above satisfies axioms (R1) to (R4) of *Proposition 4.2*. In fact the proofs are quite similar to those of left-composition, and can be found in [Met08b].

Finally, in the situation



a whiskering operation may be defined if the following equation holds:
(LR5)

$$(N \circ \alpha) \circ L = N \circ (\alpha \circ L)$$

Proof. Let objects b_0, b_0' of $\mathbb{B}$ be given. Then the following follows immediately from definitions

$$[(N \bullet^0 \alpha) \bullet^0 L]_{b_0} = L([N \bullet^0 \alpha]_{b_0}) = L(\alpha_{Nb_0}) = [\alpha \bullet^0 L]_{Nb_0} = [N \circ (\alpha \circ L)]_{b_0}$$

Analogously, consider:

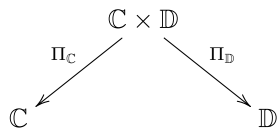
$$\begin{aligned}
[(N \bullet^0 \alpha) \bullet^0 L]_1^{b_0,b_0'} &= [N \bullet^0 \alpha]_1^{b_0,b_0'} \bullet^0 L_1^{F(Nb_0),G(Nb_0')} \\
&= \left(N_1^{b_0,b_0'} \bullet^0 \alpha_1^{Nb_0,Nb_0'}\right) \bullet^0 L_1^{F(Nb_0),G(Nb_0')} \\
&\overset{(1)}{=} N_1^{b_0,b_0'} \bullet^0 \left(\alpha_1^{Nb_0,Nb_0'} \bullet^0 L_1^{F(Nb_0),G(Nb_0')}\right) \\
&= N_1^{b_0,b_0'} \bullet^0 [\alpha \bullet^0 L]_1^{Nb_0,Nb_0'} \\
&= [N \bullet^0 (\alpha \bullet^0 L)]_1^{b_0,b_0'}
\end{aligned}$$

where everything comes directly from definitions, but (1) that is exactly the whiskering in $(n-1)$-Cat. □

Products in nCat: 2-universality of categorical products

In order to close the induction on the definition of nCat, all we need is to show that it admits finite products, according to the 2-dimensional *Universal Property 4.7*, i.e. to show it admits binary products and terminal objects. Here we give just an idea of the proof.

Let two n-categories $\mathbb{C}$ and $\mathbb{D}$ be given. We know from the discussion above that the underlying category $\lfloor n\text{−Cat} \rfloor$ admits a (standard) product of $\mathbb{C}$ and $\mathbb{D}$:



Now suppose we are given two 2-morphisms

$$\alpha : A \Rightarrow A' : \mathbb{X} \to \mathbb{C} \times \mathbb{D}, \quad \beta : B \Rightarrow B' : \mathbb{X} \to \mathbb{C} \times \mathbb{D}$$

According to *Universal Property 4.7*, what we want to prove is that there exists a unique 2-morphism $\theta : T \Rightarrow T' : \mathbb{X} \to \mathbb{C} \times \mathbb{D}$ such that

$$\theta \bullet^0 \Pi_{\mathbb{C}} = \alpha, \quad \theta \bullet^0 \Pi_{\mathbb{D}} = \beta, \tag{4}$$

In fact T and T' are determined by the 1-dimensional universal property: T is such that (and univocally determined by) $\begin{cases} T \bullet^0 \Pi_{\mathbb{C}} = A \\ T \bullet^0 \Pi_{\mathbb{D}} = B \end{cases}$, T' is such that (and univocally determined by) $\begin{cases} T' \bullet^0 \Pi_{\mathbb{C}} = A' \\ T' \bullet^0 \Pi_{\mathbb{D}} = B' \end{cases}$.

The 2-morphism $\theta = \langle \theta_0, \theta_1 \rangle$ is given by letting $\theta_0(x_0) = (\alpha_0(x_0), \beta_0(x_0))$ and $\theta_1^{x_0,x_0'} = \langle \alpha_1^{x_0,x_0'}, \beta_1^{x_0,x_0'} \rangle$.

Let us observe that in order to guarantee the compatibility of the definition w.r.t. domains and co-domains, and in order to show that the pair $\langle \theta_0, \theta_1 \rangle$ is indeed a 2-morphism of n-categories, the 2-universal property of products in $(n-1)$-Cat must be used.

3 The sesqui^2–categorical structure of n-Cat

3.1 Lax n-modification

So far we have shown that n-categories organizes naturally into a sesqui-category. This gives a setting to deal not only with n-categories and n-functors, but also with their 2-morphisms, namely lax-n-transformations.

Yet the necessity of dealing with 3-morphisms (lax-n-modifications) is the reason why we have introduced the new concept of *sesqui*^{2}*-category*, as detailed in *Appendix 4.3*. In fact, most of the theory relies on the 2-dimensional setting provided by the sequi-categorical structure developed in the previous sections. Nevertheless a notion of 3-morphism will be the main tool in giving the pullback construction in n-**Cat** a good behaviour with respect to its sesqui-categorical structure. Hence the rest of the section is devoted to give a proof of the following

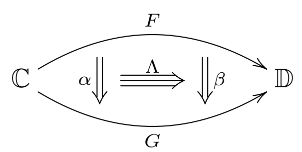
Theorem 3.1. *The sesqui-category nCat, endowed with 3-morphism, their compositions, whiskering and dimension raising 0-composition of 2-morphisms is a sesqui*^{2}*-category.*

This is done by means of the characterization given in *Theorem 4.14*.

As usual the approach is genuinely inductive, starting with the well known definition of a *modification* in Cat [Bor94].

Hence suppose given an integer $n > 1$.

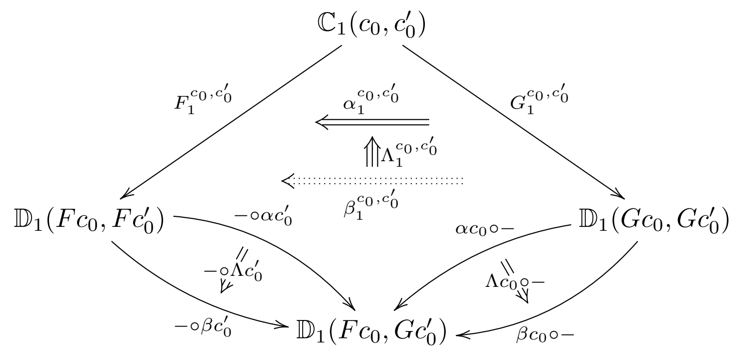
A *lax n-modification* $\Lambda : \alpha \Rrightarrow \beta : F \Longrightarrow G : \mathbb{C} \longrightarrow \mathbb{D}$



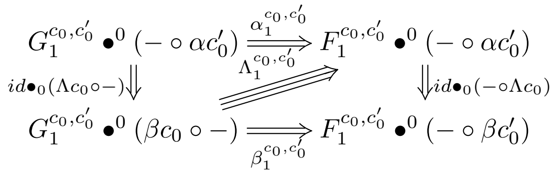
is a pair $\langle \Lambda_0, \Lambda_1 \rangle$, where

- $\Lambda_0 : \mathbb{C}_0 \longrightarrow \coprod_{c_0 \in \mathbb{C}_0} [\mathbb{D}_2(\alpha_0(c_0), \beta_0(c_0))]_0$ is a map such that, for every c_0 in $\mathbb{C}_0$, $\Lambda_0(c_0) : \alpha_0(c_0) \Longrightarrow \beta_0(c_0)$.

• (*n-naturality*) for every pair of objects c_0, c_0' of $\mathbb{C}$, a 3-morphism of $(n-1)$categories that *fills* the following diagram:



i.e.



These data must obey to *functoriality* axioms described by the following equations in (n-1)-Cat:
• (*functoriality w.r.t. 0-composition*) for every triple c_0, c_0', c_0'' of objects of $\mathbb{C}$

$$(\Lambda_1^{c_0,c_0'} \circ G_1^{c_0',c_0''}) \bullet^2 (F_1^{c_0,c_0'} \circ \Lambda_1^{c_0',c_0''}) = (- \circ^{c_0,c_0',c_0''} -) \bullet^0 \Lambda_1^{c_0,c_0''}$$

where the 2-dimensional intersection is the 2-morphism $F(-) \circ \Lambda c_0' \circ G(-)$.

• (*functoriality w.r.t. units*) for every object c_0 of $\mathbb{C}$

$$u(c_0) \bullet^0 \Lambda_1^{c_0,c_0} = Id_{[\Lambda c_0]}$$

We write $[\Lambda c_0]$ for the constant 2-morphism given by Λc_0.
Notice that both functoriality axioms for 3-morphisms reduce to those for 2-morphisms, when we consider only identity 3-morphisms (i.e. 2-morphisms *considered as* 3-morphisms).
In the same way functoriality axioms for 2-morphisms reduce to those for 1-morphisms, when we consider only identity 3-morphisms (i.e. 2-morphisms *considered as* 3-morphisms).

3.2 n-**Cat**$(\mathbb{C}, \mathbb{D})$: the underlying category

Here and in the following three small sections we consider n-categories $\mathbb{C}$ and $\mathbb{D}$ be given. We consider a sesqui-category structure over the category n-Cat$(\mathbb{C}, \mathbb{D})$. As we did in defining the sesqui-category nCat, we start by showing the underlying category structure. This has been already detailed in section 2.2, hence it suffices to recall that:

- objects of $\lfloor n\text{-Cat}(\mathbb{C}, \mathbb{D})\rfloor$ are n-functors $\mathbb{C} \to \mathbb{D}$;
- arrows of $\lfloor n\text{-Cat}(\mathbb{C}, \mathbb{D})\rfloor$ n-lax transformation between them.

Composition is 2-morphisms 1-composition, obvious units.

3.3 n-Cat$(\mathbb{C},\mathbb{D})$: the hom-categories

Let us fix n-functors $F, G : \mathbb{C} \to \mathbb{D}$. We have to define categories $\big(n\text{-Cat}(\mathbb{C},\mathbb{D})\big)(F,G)$, or more simply n-Cat(F,G).

- Objects of n-Cat(F,G) are 2-morphisms $\alpha : F \Rightarrow G$;
- Arrows $\alpha \to \beta$ are 3-morphisms of n-categories.

Composition

For 3-morphisms $\Lambda = (\Lambda_0, \Lambda_1^{-,-}) : \alpha \to \beta$ and $\Sigma = (\Sigma_0, \Sigma_1^{-,-}) : \beta \to \gamma$ their 2-composition $\Lambda \bullet^2 \Sigma : \alpha \to \gamma$ is given by the following data:

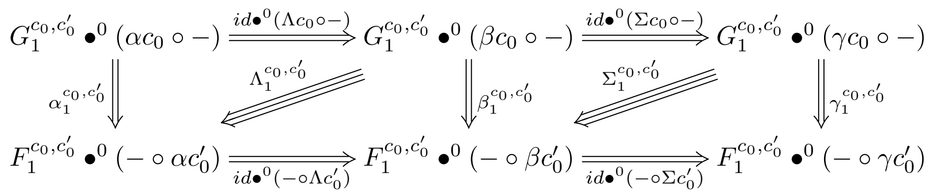
- (*on objects*)

$$[\Lambda \bullet^2 \Sigma]_0 : c_0 \mapsto \Lambda c_0 \circ^1 \Sigma c_0$$

- (*on homs*) For chosen objects c_0, c_0' one has

$$[\Lambda \bullet^2 \Sigma]_1^{c_0,c_0'} = \Big(\big(G_1^{c_0,c_0'} \bullet^0 (\Lambda c_0 \circ -)\big) \bullet^1 \Sigma_1^{c_0,c_0'} \Big) \bullet^2 \Big(\Lambda_1^{c_0,c_0'} \bullet^1 \big(F_1^{c_0,c_0'} \bullet^0 (- \circ \Sigma c_0')\big) \Big)$$

We can represent this also as a 2-dimensional pasting, sometimes useful in proofs:



These data form indeed a 3-morphism, as the clever reader can personally check by 2-dimensional diagram chasing: in fact only 2-morphisms and 3-morphisms enter into the proof, that uses product interchange rules in dimension $n-1$.

Units

For any 2-morphism $\beta : \ F \Rightarrow G : \ \mathbb{C} \to \mathbb{D}$ its identity 3-morphisms id_β is given by:

- (*on objects*)

$$[id_\beta]_0 : c_0 \mapsto id_{\beta c_0}$$

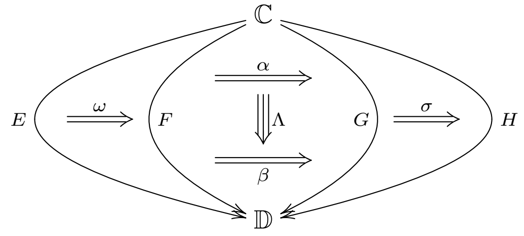
- (*on homs*) For chosen objects c_0, c_0' one has

$$[id_\beta]_1^{-,-} = id_{\beta_1^{-,-}}$$

It is immediate to check that above pair is indeed a 3-morphisms.
Similarly associativity and neutral units follows from same properties for 2-cells.

3.4 n-**Cat**$(\mathbb{C}, \mathbb{D})$: the sesqui-categorical structure

In the this section we will show that hom-categories $n-\mathrm{Cat}(\mathbb{C}, \mathbb{D})$ underly a structure of sesqui-categories, with 2-cells provided by 3-morphisms of n-categories. To this end we define reduced left/right 1-composition of a 3-morphism with a 2-morphism, according to the following reference diagram.



Reduced left- and right-composition

The 3-morphism $\omega \bullet^1 \Lambda : \ \omega \bullet^1 \alpha \Rrightarrow \omega \bullet^1 \beta$ is defined for c_0, c_0' in $\mathbb{C}$ by

$$
\begin{aligned}
[\omega \bullet^1 \Lambda]_0(c_0) &= \omega_{c_0} \circ^0 \Lambda_{c_0} \\
[\omega \bullet^1 \Lambda]_1^{c_0,c_0'} &= \left(\Lambda_1^{c_0,c_0'} \bullet^0 (\omega c_0 \circ -)\right) \bullet^1 \left(\omega_1^{c_0,c_0'} \bullet^0 (- \circ \beta c_0')\right) \\
&= (\omega c_0 \circ \Lambda_1^{c_0,c_0'}) \bullet^1 (\omega_1^{c_0,c_0'} \circ \beta c_0')
\end{aligned}
$$

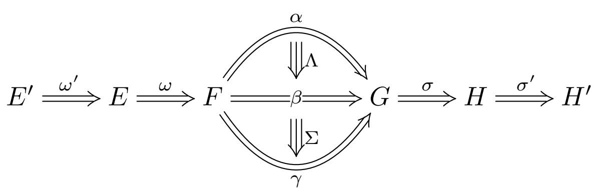
The pair $\langle [\omega \bullet^1 \Lambda]_0, [\omega \bullet^1 \Lambda]_1^{-,-} \rangle$ forms indeed a 3-morphism of n-categories. In fact it satisfies composition and unit axioms, as one can prove by induction with a consistent use of the characterization of sesqui^2-categories given in Theorem 4.14. Similarly, the 3-morphism $\Lambda \bullet^1 \sigma : \ \alpha \bullet^1 \sigma \Rrightarrow \beta \bullet^1 \sigma$ is defined for c_0, c_0' in $\mathbb{C}$ by

$$
\begin{aligned}
[\Lambda \bullet^1 \sigma]_0(c_0) &= \Lambda_{c_0} \circ^0 \sigma_{c_0} \\
[\Lambda \bullet^1 \sigma]_1^{c_0,c_0'} &= \left(\sigma_1^{c_0,c_0'} \bullet^0 (\alpha c_0 \circ -)\right) \bullet^1 \left(\Lambda_1^{c_0,c_0'} \bullet^0 (- \circ \sigma c_0')\right) \\
&= (\alpha c_0 \circ \sigma_1^{c_0,c_0'}) \bullet^1 (\Lambda_1^{c_0,c_0'} \circ \sigma c_0')
\end{aligned}
$$

The pair $\langle [\Lambda \bullet^1 \sigma]_0, [\Lambda \bullet^1 \sigma]_1^{-,-} \rangle$ forms indeed a 3-morphism of n-categories. The proof is a straightforward variation of the proof for reduced right-composition above.

Next Proposition gives some properties of left/right 1-composition of a 3-morphism with a 2-morphism (for a proof, the reader is addressed to [Met08b]). They are modeled on similar properties given in the definition of a sesqui-category, and they are extremely useful in dealing with calculations.

Proposition 3.2. *(2-composition (i.e. vertical) composition of 3-morphisms w.r.t. (reduced) 1-composition with a 2-morphism) In the situation described by the diagram below, the following equations hold:*

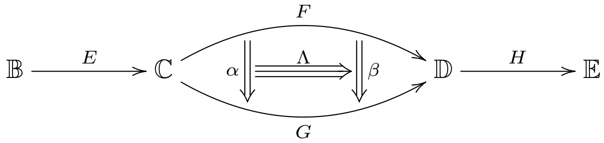


$(L1)'$	$id_F \bullet^1_L \Lambda = \Lambda$	$(R1)'$	$\Lambda \bullet^1_R id_G = \Lambda$
$(L2)'$	$(\omega' \bullet^1 \omega) \bullet^1_L \Lambda = \omega' \bullet^1_L (\omega \bullet^1_L \Lambda)$	$(R2)'$	$\Lambda \bullet^1_R (\sigma \bullet^1 \sigma') = (\Lambda \bullet^1_R \sigma) \bullet^1_R \sigma'$
$(L3)'$	$\omega \bullet^1_L id_\alpha = id_{\omega\alpha}$	$(R3)'$	$id_\alpha \bullet^1_R \sigma = id_{\alpha\sigma}$
$(L4)'$	$\omega \bullet^1_L (\Lambda \bullet^2 \Sigma) = (\omega \bullet^1_L \Lambda) \bullet^2 (\omega \bullet^1_L \Sigma)$	$(R4)'$	$(\Lambda \bullet^2 \Sigma) \bullet^1_R \sigma = (\Lambda \bullet^1_R \sigma) \bullet^2 (\Sigma \bullet^1_R \sigma)$

$$(LR5)' \quad (\omega \bullet^1_L \Lambda) \bullet^1_R \sigma = \omega \bullet^1_L (\Lambda \bullet^1_R \sigma)$$

3.5 0-whiskering of 3-morphisms

In this section we define reduced left/right 1-composition of a 3-morphism with a 1-morphism, according to the following reference diagram.



Reduced left and right-composition

The 3-morphism $E \bullet^0 \Lambda : \; E \bullet^0 \alpha \Rrightarrow E \bullet^0 \beta$ is defined for b_0, b_0' in $\mathbb{B}$ by

$$[E \bullet^0 \Lambda]_0(b_0) = \Lambda(Eb_0)$$

$$[E \bullet^0 \Lambda]_1^{b_0,b_0'} = E_1^{b_0,b_0'}(-) \bullet^0 \Lambda_1^{Eb_0,Eb_0'}$$

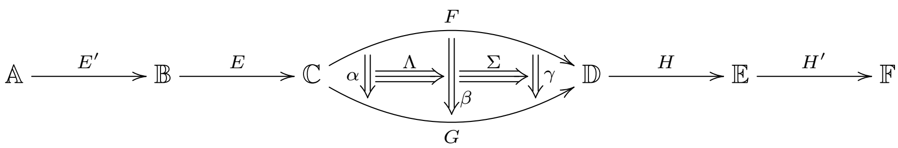
The pair $\langle [E \bullet^0 \alpha]_0, [E \bullet^0 \alpha]_1^{-,-} \rangle$ forms indeed a 3-morphism of n-categories (see [Met08b]).
Similarly the 3-morphism $\Lambda \bullet^0 H : \; \alpha \bullet^0 \Rrightarrow \beta \bullet^0 H$ is defined for c_0, c_0' in $\mathbb{C}$ by

$$[\Lambda \bullet^0 H]_0(c_0) = H(\Lambda c_0)$$

$$[\Lambda \bullet^0 H]_1^{c_0,c_0'} = \Lambda_1^{c_0,c_0'} \bullet^0 H_1^{Fc_0,Gc_0'}$$

The pair $\langle [\Lambda \bullet^0 H]_0, [\Lambda \bullet^0 H]_1^{-,-} \rangle$ forms indeed a 3-morphism of n-categories (see [Met08b]).
As we did in describing the sesqui-categorical structure for homs in nCat, we use again a *left-and-right* approach to describe properties of the 0-whiskering of of a 3-morphism with a morphism. This is done in the next statement, a proof can be found in [Met08b].

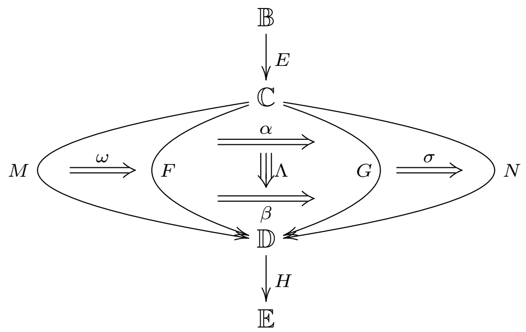
Proposition 3.3 (2-composition (i.e. vertical) composition of 3-morphisms w.r.t. (reduced) 0-composition with a (1-)morphism)**.** *In the situation described by the diagram below, the following equations hold:*



$(L1)''$	$id_{\mathbb{C}} \bullet^0_L \Lambda = \Lambda$	$(R1)''$	$\Lambda \bullet^0_R id_{\mathbb{D}} = \Lambda$
$(L2)''$	$(E' \bullet^0_L E) \bullet^0_L \Lambda = E' \bullet^0_L (E \bullet^0_L \Lambda)$	$(R2)''$	$\Lambda \bullet^0_R (H \bullet^0_R H') = (\Lambda \bullet^0_R H) \bullet^0_R H'$
$(L3)''$	$E \bullet^0_L id_\alpha = id_{E \bullet^0_L \alpha}$	$(R3)''$	$id_\alpha \bullet^0_R H = id_{\alpha \bullet^0_R H}$
$(L4)''$	$E \bullet^0_L (\Lambda \bullet^2 \Sigma) = (E \bullet^0_L \Lambda) \bullet^2 (E \bullet^0_L \Sigma)$	$(R4)''$	$(\Lambda \bullet^2 \Sigma) \bullet^0_R H = (\Lambda \bullet^0_R H) \bullet^2 (\Sigma \bullet^0_R H)$

$$(LR5)'' \quad (E \bullet_L^0 \Lambda) \bullet_R^0 H = E \bullet_L^0 (\Lambda \bullet_R^0 H)$$

Before switching to next section, let us give a last property that express at once functoriality of left and right 0-composition with a morphism. To this end, let us be given also 2-morphisms $\omega : \; M \Rightarrow F$ and $\sigma : \; G \Rightarrow N$, as represented in the diagram below



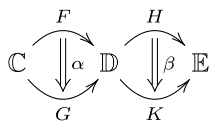
Left/right 0-composition of a 3-morphism with a morphism satisfies also the following property that relates 0-whiskering w.r.t. 1-whiskering:

Proposition 3.4 (Whiskering interchange property)**.**

$$(LRW) \qquad E \bullet_L^0 \big(\omega \bullet_L^1 \Lambda \bullet_R^1 \sigma\big) \bullet_R^0 H = (E \bullet_L^0 \omega \bullet_R^0 H) \bullet_L^1 (E \bullet_L^0 \Lambda \bullet_R^0 H) \bullet_R^1 (E \bullet_L^0 \sigma \bullet_R^0 H)$$

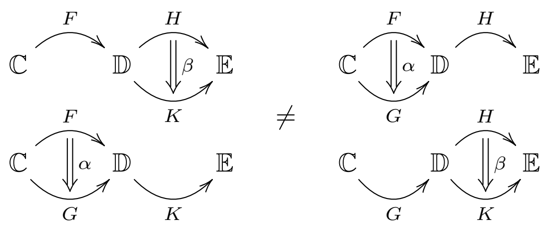
3.6 Dimension raising 0-composition of 2-morphisms

Let two 0-intersecting 2-morphisms of n-categories be given.



It is easy to verify that in general

$$\alpha \setminus \beta := (F \bullet^0 \beta) \bullet^1 (\alpha \bullet^0 K) \neq (\alpha \bullet^0 H) \bullet^1 (G \bullet^1 \beta) =: \alpha / \beta$$

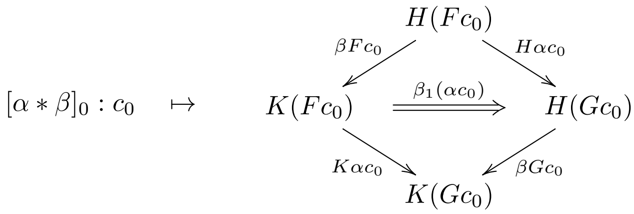


(5)

Indeed they give the data for a 3-morphism

$$\alpha * \beta : \alpha \setminus \beta \Longrightarrow \alpha / \beta$$

In fact, for every object c_0 of $\mathbb{C}$ one defines



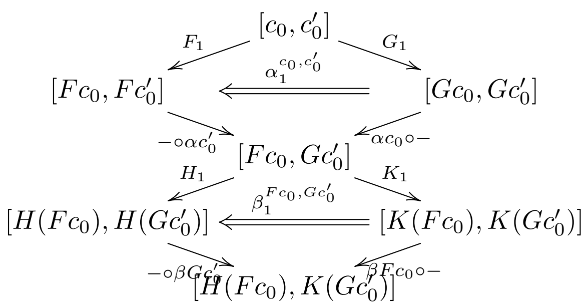
Moreover for every pair of objects c_0, c_0' of $\mathbb{C}$ one defines

$$[\alpha * \beta]_1^{c_0,c_0'} = \alpha_1^{c_0,c_0'} * \beta_1^{Fc_0,Gc_0'}$$

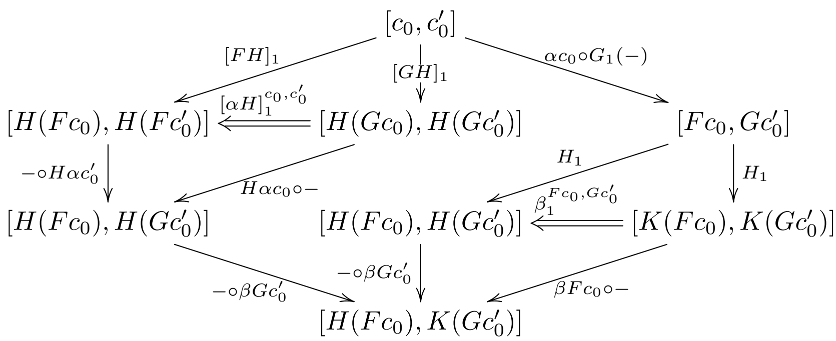
A detailed proof that the pair $\langle [\alpha * \beta]_0, [\alpha * \beta]_1^{-,-} \rangle$ satisfies the axioms for a 3-morphism can be found in [Met08b]. It is quite long and involves some 3-dimensional diagram-chasing and sesqui^2–categorical properties. Here we will be content to persuade the reader that domain and codomain of $[\alpha * \beta]_1^{c_0,c_0'}$ are well defined.
First we write the diagram that represents the 3-morphism of (n-1)categories

$$[\alpha * \beta]_1^{c_0,c_0'} : \ \alpha_1^{c_0,c_0'} \setminus \beta_1^{Fc_0,Gc_0'} \Longrightarrow \alpha_1^{c_0,c_0'} / \beta_1^{Fc_0,Gc_0'}$$

i.e. the composition



Its domain is computed below



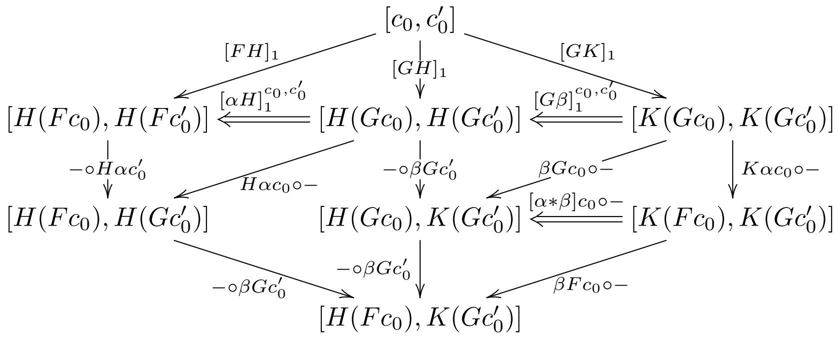
Now, by functoriality w.r.t 0-composition, with constant left composite one has

$$(\alpha c_0 \circ -) \bullet^0 \beta_1^{Fc_0,Gc_0'} = \Big(K_1^{Gc_0,Gc_0'} \bullet^0 (\beta_1(\alpha c_0) \circ -) \Big) \bullet^1 \Big(\beta_1^{Gc_0,Gc_0'} \bullet^0 (H\alpha c_0 \circ -) \Big)$$

and by definition of $*$-composition on objects,

$$= \left(K_1^{Gc_0, Gc_0'} \bullet^0 ([\alpha * \beta] c_0 \circ -) \right) \bullet^1 \left(\beta_1^{Gc_0, Gc_0'} \bullet^0 (H\alpha c_0 \circ -) \right)$$

Hence we can redraw the domain

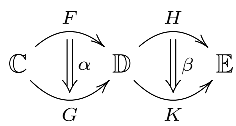


And this completes the domain-part. Concerning the codomain, the calculation is similar and it is left to the reader.

Remark 3.5. We have adopted the $*$-symbol instead of the more obvious $\bullet^0$ in order to emphasize the dimension-raising property of this composition. Nevertheless $*$-properties w.r.t. other $\bullet^0$-compositions are somehow better understood thinking only in terms of $\bullet^0$.

The following statements give some properties of dimension raising composition of 2-morphisms. The proofs inductively relies on the similar properties in lower dimension, and can be found in [Met08b].

Proposition 3.6. *Given the case*

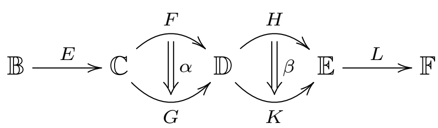


If α is a lax natural n-transformation and β is a strict natural n-transformation, the composition $\alpha * \beta$ is an identity.
In this case it is possible to deal with dimension preserving 0-composition of 2-morphisms, by letting

$$\alpha \tilde{*} \beta = \mathrm{dom}(\alpha * \beta) = \mathrm{cod}(\alpha * \beta)$$

Importance of Proposition above is in that it allows to right-0-compose freely with constant transformations, such as $- \circ c_2$ or $c_2 \circ -$ for a 2-cell $c_2 : c_1 \Rightarrow c_1' : c_0 \to c_0'$.Notice that it does not hold for α strict and β lax, since in this case the result is a strict 3-morphism.

Let be given the situation



one has the following

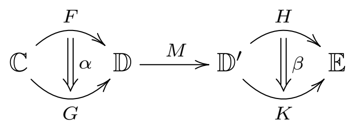
Proposition 3.7 ($*$-associativity 1)**.**

$$(L * A) \quad (E \bullet^0_L \alpha) * \beta = E \bullet^0_L (\alpha * \beta) \quad (R * A) \quad \alpha * (\beta \bullet^0_R L) = (\alpha * \beta) \bullet^0_R L$$

Proposition 3.8 ($*$-identity)**.**

$$(L) \quad id_E * \alpha = id_{E \bullet^0_L \alpha} \quad (R) \quad \alpha * id_H = id_{\alpha \bullet^0_R H}$$

In the situation

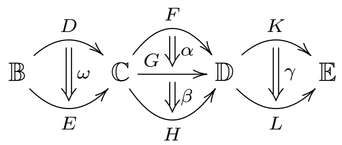


one has the following

Proposition 3.9 ($*$-associativity 2)**.**

$$\alpha * (M \bullet^0_L \beta) = (\alpha \bullet^0_R M) * \beta$$

In the situation below



one has the following

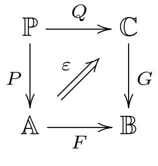
Proposition 3.10 ($*$-functoriality)**.**

$$(a) \qquad (\alpha \bullet^1 \beta) * \gamma = \Big((\alpha * \gamma) \bullet^1 (\beta \bullet^0 L) \Big) \bullet^2 \Big((\alpha \bullet^0 K) \bullet^1 (\beta * \gamma) \Big)$$

$$(b) \qquad \omega * (\alpha \bullet^1 \beta) = \Big((\omega * \alpha) \bullet^1 (E \bullet^0 \beta) \Big) \bullet^2 \Big((D \bullet^0 \alpha) \bullet^1 (\omega * \beta) \Big)$$

3.7 h-Pullbacks revisited: h^2-pullbacks in n-Cat

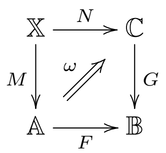
We introduce here a notion of 2-dimensional h-pullback in the sesqui^2-category n-Cat. Indeed the notion of h^2-pullback can be formulated in any sesqui^2-category, and it is easy to show that h^2-pullbacks satisfy trivially the 1-dimensional universal property (see Proposition 3.12), hence they are also h-pullbacks. For instance, our construction of the *standard* h-pullback of n-categories is an instance of such a 2-dimensional one.
In order to fix notation, let us consider two n-functors $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{C} \to \mathbb{B}$. A h^2-pullback of F and G is a four-tuple $(\mathbb{P}, P, Q, \varepsilon)$



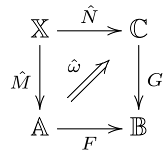
that satisfies the following 2-dimensional universal property:

Universal Property 3.11 (h^2-pullbacks)**.** *For any other two four-tuple*

$(\mathbb{X}, M, N, \omega)$ $\qquad\qquad$ $(\mathbb{X}, \hat{M}, \hat{N}, \hat{\omega})$

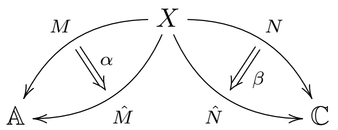


and

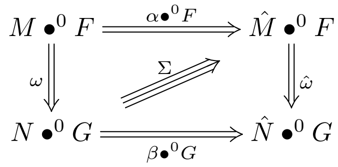


$3-$*morphism* Σ

$2-$*morphism* α, β



and



there exists a unique $\lambda: \ L \Rightarrow \hat{L}: \ \mathbb{X} \to \mathbb{P}$ *such that (UP)*

$$1.\ \lambda \bullet^0 P = \alpha, \quad 2.\ \lambda \bullet^0 Q = \beta, \quad 3.\ \lambda * \varepsilon = \Sigma.$$

As an immediate consequence of the definition, we state the following

Proposition 3.12. 2-Universal Property *of* h^2*-pullbacks implies* 1*-dimensional one. Hence* h^2*-pullbacks are defined up to isomorphism.*

Proof. Just put α, β and Σ identities. $\square$

Let us notice that *Proposition 3.12* holds in every sesqui^2-category. More interestingly in n-Cat a kind of converse to this proposition also holds.

Theorem 3.13. *The sesqui*^{2}*-category* n*-Cat admits* h^2*-pullbacks. In fact given two* n*-functors* $F : \mathbb{A} \to \mathbb{B}$*,* $G : \mathbb{C} \to \mathbb{B}$*, their standard* h*-pullback* $(\mathbb{P}, P, Q, \varepsilon)$ *satisfies also* Universal property 3.11.

Proof. Firstly we remark that 1-dimensional *Universal Property 4.10* of h-pullbacks applied to the four-tuple $(\mathbb{X}, M, N, \omega)$ yields an $L : \mathbb{X} \to \mathbb{P}$, while applied to $(\mathbb{X}, \hat{M}, \hat{N}, \hat{\omega})$, a $\hat{L} : \mathbb{X} \to \mathbb{P}$. Those have to be *domain* and *co-domain* of the 2-cell provided by the universal property, namely $\lambda: \ L \Rightarrow \hat{L}$.
We recall the constructions in order to fix notation.
For x_0, x_0' objects of $\mathbb{X}$, L_0 is defined by $L_0(x_0) = (Mx_0, \omega_{x_0}, Nx_0)$, while $L_1^{x_0,x_0'}$ is given by the universal property in dimension $n-1$, i.e. it is the unique morphism $\mathbb{X}_1(x_0, x_0') \to \mathbb{P}_1(L(x_0), L(x_0'))$ such that $L_1^{x_0,x_0'} \bullet^0 P_1^{Lx_0,Lx_0'} = M_1^{x_0,x_0'}, L_1^{x_0,x_0'} \bullet^0 Q_1^{Lx_0,Lx_0'} = N_1^{x_0,x_0'}, L_1^{x_0,x_0'} \bullet^0 \varepsilon_1^{Lx_0,Lx_0'} = \omega_1^{x_0,x_0'}$ The pair $L = (L_0, L_1^{-,-})$ is a 1-morphism. Similarly one determines $\hat{L} = (\hat{L}_0, \hat{L}_1^{-,-})$.
Now we show that remaining data (namely, α, β and Σ) of the hypothesis provide a 2-morphism $\lambda : L \Rightarrow \hat{L}$ that satisfies required property. In fact the object-part is defined directly

$$\lambda_{x_0} = (\alpha_{x_0}, \Sigma_{x_0}, \beta_{x_0}): \ Lx_0 \to \hat{L}x_0,$$

while $\lambda_1^{x_0,x-0'}$

$$\begin{array}{ccccc} & & \mathbb{X}_1(x_0,x'_0) & & \\ & \swarrow^{L_1^{x_0,x'_0}} & & \searrow^{\hat{L}_1^{x_0,x'_0}} & \\ \mathbb{P}_1(Lx_0,Lx'_0) & & \overset{\lambda_1^{x_0,x'_0}}{\Longleftarrow} & & \mathbb{P}_1(\hat{L}x_0,\hat{L}x'_0) \\ & \searrow_{-\circ\lambda x'_0} & & \swarrow_{\lambda x_0\circ-} & \\ & & \mathbb{P}_1(Lx_0,\hat{L}x'_0) & & \end{array}$$

is given by the 2-universal property for (n-1)categories. In fact the 0-codomain of $\lambda_1^{x_0,x'_0}$, namely $\mathbb{P}_1(Lx_0,\hat{L}x'_0)$ is defined inductively as a h^2-pullback in (n-1)-Cat:

$$\begin{array}{ccccc} \mathbb{P}_1(Lx_0,\hat{L}x'_0) & & \xrightarrow{Q_1^{Lx_0,\hat{L}x'_0}} & & \mathbb{C}_1(Nx_0,\hat{N}x'_0) \\ & & & & \downarrow G_1^{Nx_0,\hat{N}x'_0} \\ \downarrow P_1^{Lx_0,\hat{L}x'_0} & & \overset{\varepsilon_1^{Lx_0,\hat{L}x'_0}}{\Longleftarrow} & & \mathbb{B}_1(G(Nx_0),G(\hat{N}x'_0)) \\ & & & & \downarrow \omega x_0\circ- \\ \mathbb{A}_1(Mx_0,\hat{M}x'_0) & \xrightarrow[F_1^{Mx_0,\hat{M}x'_0}]{} & \mathbb{B}_1(F(Mx_0),F(\hat{M}x'_0)) & \xrightarrow[-\circ\hat{\omega}x'_0]{} & \mathbb{B}_1(F(Mx_0),G(\hat{N}x'_0)) \end{array}$$

Over the same base are also defined

$$\begin{array}{ccccc} \mathbb{X}_1(x_0,x'_0) & & \xrightarrow{N_1^{x_0,x'_0}\circ\beta x'_0} & & \mathbb{C}_1(Nx_0,\hat{N}x'_0) \\ & & & & \downarrow G_1^{Nx_0,\hat{N}x'_0} \\ \downarrow M_1^{x_0,x'_0}\circ\alpha x'_0 & & \theta \;=\; (\omega_1^{x_0,x'_0}\circ G(\beta x'_0))\bullet^1([MF]_1^{x_0,x'_0}\circ\Sigma x'_0) \;\Longleftarrow & & \mathbb{B}_1(G(Nx_0),G(\hat{N}x'_0)) \\ & & & & \downarrow \omega x_0\circ- \\ \mathbb{A}_1(Mx_0,\hat{M}x'_0) & \xrightarrow[F_1^{Mx_0,\hat{M}x'_0}]{} & \mathbb{B}_1(F(Mx_0),F(\hat{M}x'_0)) & \xrightarrow[-\circ\hat{\omega}x'_0]{} & \mathbb{B}_1(F(Mx_0),G(\hat{N}x'_0)) \end{array}$$

and

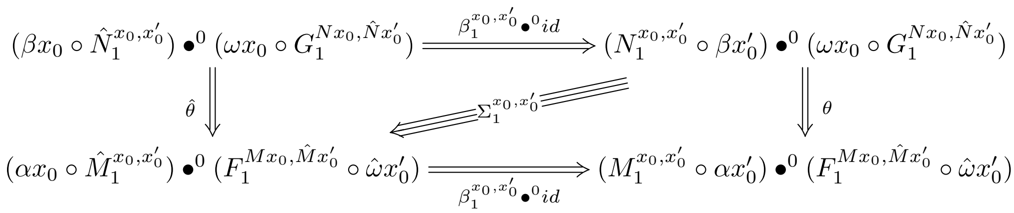
$$\begin{array}{ccccc} \mathbb{X}_1(x_0,x'_0) & & \xrightarrow{\beta x_0\circ\hat{N}_1^{x_0,x'_0}} & & \mathbb{C}_1(Nx_0,\hat{N}x'_0) \\ & & & & \downarrow G_1^{Nx_0,\hat{N}x'_0} \\ \downarrow \alpha x_0\circ\hat{M}_1^{x_0,x'_0} & & \hat{\theta} \;=\; (\Sigma x_0\circ[\hat{N}G]_1^{x_0,x'_0})\bullet^1(F(\alpha x_0)\circ\hat{\omega}_1^{x_0,x'_0}) \;\Longleftarrow & & \mathbb{B}_1(G(Nx_0),G(\hat{N}x'_0)) \\ & & & & \downarrow \omega x_0\circ- \\ \mathbb{A}_1(Mx_0,\hat{M}x'_0) & \xrightarrow[F_1^{Mx_0,\hat{M}x'_0}]{} & \mathbb{B}_1(F(Mx_0),F(\hat{M}x'_0)) & \xrightarrow[-\circ\hat{\omega}x'_0]{} & \mathbb{B}_1(F(Mx_0),G(\hat{N}x'_0)) \end{array}$$

Moreover we can consider 2-morphisms:

$$\alpha_1^{x_0,x'_0}:\; \alpha x_0\circ\hat{M}_1^{x_0,x'_0}\Rightarrow M_1^{x_0,x'_0}\circ\alpha x'_0:\; \mathbb{X}_1(x_0,x'_0)\to\mathbb{A}_1(Mx_0,\hat{M}x'_0)$$

$$\beta_1^{x_0,x'_0}:\; \beta x_0\circ\hat{N}_1^{x_0,x'_0}\Rightarrow N_1^{x_0,x'_0}\circ\beta x'_0:\; \mathbb{X}_1(x_0,x'_0)\to\mathbb{A}_1(Mx_0,\hat{M}x'_0)$$

and the 3-morphism



Finally we can apply the universal property, in order to get a *unique* 2-morphism

$$\lambda_1^{x_0,x_0'} : \ L_1^{x_0,x_0'} \circ \lambda x_0' \Rightarrow \lambda x_0 \circ \hat{L}_1^{x_0,x_0'}$$

such that $\lambda_1^{x_0,x_0'} \bullet^0 Q_1^{Lx_0,\hat{L}x_0'} = \beta_1^{x_0,x_0'}$, $\lambda_1^{x_0,x_0'} \bullet^0 P_1^{Lx_0,\hat{L}x_0'} = \alpha_1^{x_0,x_0'}$ and $\lambda_1^{x_0,x_0'} * \varepsilon_1^{Lx_0,\hat{L}x_0'} = \Sigma_1^{x_0,x_0'}$ The proof that the pair $\lambda = < \lambda_0, \lambda_1^{-,-} >$ is a 2-morphism of n-categories is quite technical. The interested reader will find it in [Met08b], *Lemma 6.4*.
Moreover it satisfies by construction *Universal Property 3.11.* Finally

$$[\lambda * \varepsilon]_{x_0} = \varepsilon(\lambda_{x_0}) = \varepsilon\big((\alpha_{x_0}, \Sigma_{x_0}, \beta_{x_0})\big) = \Sigma_{x_0}$$

and

$$[\lambda * \varepsilon]_1^{x_0,x_0'} = \lambda_1^{x_0,x_0'} * \varepsilon_1^{Lx_0,\hat{L}x_0'} = \Sigma_1^{x_0,x_0'}$$

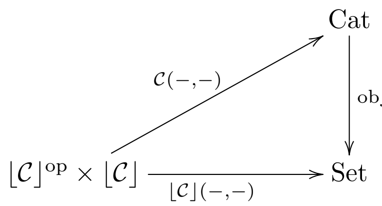
To conclude the proof we still need to prove uniqueness. But this will easily be achieved. Indeed the object part of 2-morphism λ satisfying the universal property is univocally determined by the fact that P_0, Q_0 and ε_0 are projection, and once that is determined, uniqueness in dimension $n-1$ guaranties the homs part. □

4 Higher dimensional structures

4.1 Sesqui-categories, their morphisms and 2-morphisms

The notion of sesqui-category is due to Ross Street [Str96]. More recent developing can be found in [MF08]. The term *sesqui* comes from the latin *semis-que*, that means (one and) a half. Hence a sesqui-category is something in-between a category and a 2-category. More precisely

Definition 4.1. *A (small) sesqui-category $\mathcal{C}$ is a (small) category $\lfloor \mathcal{C} \rfloor$ with a lifting of the hom-functor to Cat, such that the following diagram of categories and functors commutes,* obj *being the functor that forgets the morphisms:*



(6)

Objects and morphisms of $\lfloor \mathcal{C} \rfloor$ are also objects and 1-cells of $\mathcal{C}$, while morphisms of $\mathcal{C}(A,B)$'s (with A and B running in $\mathrm{obj}(\lfloor \mathcal{C} \rfloor)$*) are the 2-cells of $\mathcal{C}$.*

We first observe that the definition above induces a 2-graph structure on $\mathcal{C}$, whose underlying graph underlies the category $\mathcal{C}$. Besides, the functor $\mathcal{C}(-,-)$ provides hom-sets of the category $\lfloor\mathcal{C}\rfloor$ with a category structure, whose composition is termed *vertical composition* (or 1-composition) of 2-cells. Finally, condition expressed by diagram (6) on the lifting $\mathcal{C}(-,-)$ gives a reduced horizontal composition, or *whiskering* (or 0-composition), compatible with 1-cell composition and with the 2-graph structure of $\mathcal{C}$. In fact, for $A' \xrightarrow{a} A$ and $B \xrightarrow{b} B'$ in $\lfloor\mathcal{C}\rfloor$, the functor

$$\mathcal{C}(a,b) : \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A',B')$$

gives explicitly such a composition: for a 2-cell $\alpha : f \Longrightarrow g : A \rightarrow B$, it whiskers the diagram

$$A' \xrightarrow{a} A \underset{g}{\overset{f}{\Downarrow \alpha}} B \xrightarrow{b} B'$$

to get the 2-cell

$$A' \underset{a\bullet g\bullet b}{\overset{a\bullet f\bullet b}{\Downarrow a\bullet\alpha\bullet b}} B'$$

where $a\bullet\alpha\bullet b$ is just a concise form for $\mathcal{C}(a,b)(\alpha)$. By functoriality of whiskering, the operation may also be given in a *left-and-right* fashion. In fact it suffices to identify

$$a \bullet_L \alpha = a \bullet \alpha \bullet 1_B, \quad \alpha \bullet_R b = 1_A \bullet \alpha \bullet b$$

This fact can be made precise, and gives a more tractable definition, by the following characterization (see, for example [Gra94, Ste94]):

Proposition 4.2. *Let $\mathcal{C}$ be a reflexive 2-graph* $\mathcal{C}_2 \underset{t}{\overset{s}{\rightleftarrows}} \mathcal{C}_1 \underset{t}{\overset{s}{\rightleftarrows}} \mathcal{C}_0$ *whose underlying graph* $\lfloor\mathcal{C}\rfloor = \mathcal{C}_1 \underset{t}{\overset{s}{\rightleftarrows}} \mathcal{C}_0$ *has a category structure. Then $\mathcal{C}$ is a sesqui-category precisely when the following conditions hold:*

1. *for every pair of objects A, B of $\mathcal{C}_0$, the graph $\mathcal{C}(A,B)$ has a category structure, called the hom-category of A, B.*
2. *(partial) reduced horizontal compositions are defined,* i.e. *for every A', A, B and B' objects of $\mathcal{C}_0$, composition in $\lfloor\mathcal{C}\rfloor$ extends to binary operations*

$$\bullet_L \quad : \quad : \lfloor\mathcal{C}\rfloor(A',A) \times \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A',B) \tag{7}$$

$$\bullet_R \quad : \quad \mathcal{C}(A,B) \times \lfloor\mathcal{C}\rfloor(B,B') \longrightarrow \mathcal{C}(A,B') \;, \tag{8}$$

that satisfy equations below, whenever the composites are defined:

$$A'' \xrightarrow{a'} A' \xrightarrow{a} A \underset{g}{\overset{f}{\Downarrow \alpha}} B \xrightarrow{b} B' \xrightarrow{b'} A''$$

$$
\begin{array}{llll llll}
(L1) & 1_A \bullet_L \alpha & = & \alpha & (R1) & \alpha \bullet_R 1_B & = & \alpha \\
(L2) & a'a \bullet_L \alpha & = & a' \bullet_L (a \bullet_L \alpha) & (R2) & \alpha \bullet_R bb' & = & (\alpha \bullet_R b) \bullet_R b' \\
(L3) & a \bullet_L 1_f & = & 1_{af} & (R3) & 1_f \bullet_R b & = & 1_{fb} \\
(L4) & a \bullet_L (\alpha \cdot \beta) & = & (a \bullet_L \alpha) \cdot (a \bullet_L \beta) & (R4) & (\alpha \cdot \beta) \bullet_R b & = & (\alpha \bullet_R b) \cdot (\beta \bullet_R b) \\
(LR5) & (a \bullet_L \alpha) \bullet_R b & = & a \bullet_L (\alpha \bullet_R b) & & & &
\end{array}
$$

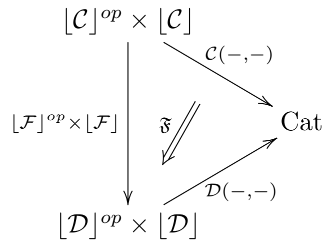
In these equations, 1_A and 1_B are identity 1-cells, while 1_f, 1_{af} and 1_{fb} are identity 2-cells, and $\cdot$ is the (vertical) composition inside the hom-categories. Axiom (LR5) will be also called whiskering axiom*.*

Morphisms between sesqui-categories are termed sesqui-functors. More precisely a sesqui-functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is a 2-graph morphism such that

- $\lfloor \mathcal{F} \rfloor : \lfloor \mathcal{C} \rfloor \longrightarrow \lfloor \mathcal{D} \rfloor$ is a functor,
- for every A, B in $\mathcal{C}_0$,

$$\mathcal{F}^{A,B} : \mathcal{C}(A, B) \longrightarrow \mathcal{D}(\mathcal{F}(A), \mathcal{F}(B))$$

are functors component of a natural transformation $\mathfrak{F}$



(9)

that lifts $\lfloor \mathfrak{F} \rfloor : \lfloor \mathcal{C} \rfloor(-,-) \Longrightarrow (\lfloor \mathcal{F} \rfloor^{op} \times \lfloor \mathcal{F} \rfloor) \cdot \lfloor \mathcal{D} \rfloor(-,-)$.

Remark 4.3. Notice that every functor between categories gives rise to such a natural transformation as $\lfloor \mathfrak{F} \rfloor$ for $\lfloor \mathcal{F} \rfloor$. From this point of view, the last condition may be re-formulated saying that *a sesqui-functor is the lifting of a functor between the underlying categories.*

We can translate the definition of sesqui-functor in terms of left/right compositions, according to the next easy to prove

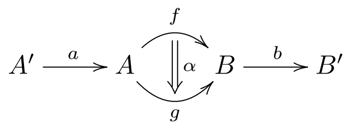
Proposition 4.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be sesqui-categories, and let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be a 2-graphs homomorphism, whose underlying graph homomorphism $\lfloor \mathcal{F} \rfloor$ is a functor. Then $\mathcal{F}$ is a sesqui-functor precisely when the following conditions hold:*

1. *for every pair of objects A, B of $\mathcal{C}_0$, the graph homomorphism*

$$\mathcal{F}^{A,B} : \mathcal{C}(A, B) \longrightarrow \mathcal{D}(\mathcal{F}(A), \mathcal{F}(B))$$

is a functor, called the hom-functor at A, B.

2. *(partial) horizontal reduced compositions are preserved,* i.e. *for every diagram*



in $\mathcal{C}_0$*, equations below hold:*

$$(L6) \quad \mathcal{F}(a \bullet_L \alpha) \; = \; \mathcal{F}(a) \bullet_L \mathcal{C}(\alpha) \qquad (R6) \quad \mathcal{F}(\alpha \bullet_R b) \; = \; \mathcal{F}(\alpha) \bullet_R \mathcal{F}(b)$$

Definition 4.5 (strict sesqui-transformations)**.** *Let two parallel sesqui-functors*

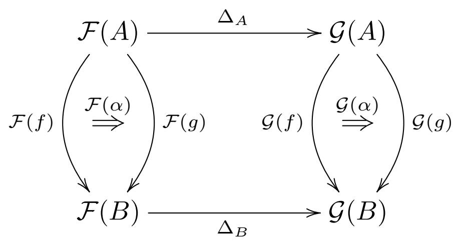
$$\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$$

be given, and let be given a 2-graph transformation $\Delta : \mathcal{F} \Rightarrow \mathcal{G}$ *whose underlying 1-transformation*

$$\lfloor \Delta \rfloor : \lfloor \mathcal{F} \rfloor \Rightarrow \lfloor \mathcal{G} \rfloor$$

is a natural transformation of functors. Then Δ *is a (strict) natural transformation of sesqui-functors when, for every* $\alpha : f \Rightarrow g : A \to B$ *in* $\mathcal{C}$*,*

$$\mathcal{F}(\alpha) \bullet_R \Delta_B = \Delta_A \bullet_L \mathcal{G}(\alpha)$$



Notice that while vertical composition of (strict) natural transformation of sesqui-functors can be easily defined, the same is not true for horizontal composition. Therefore the category SesquiCAT of sesqui-categories, regardless of size issues, is indeed a sesqui-category itself.

The notion of (strict) natural transformation of sesqui-functors is essentially of a categorical nature. Namely the "functor"

$$\lfloor - \rfloor : \text{SesquiCAT} \to \text{CAT}$$

is also a "sesqui-functor", when we consider the 2-category CAT as a sesqui-category.
Therefore those are just usual natural transformations that behave nice with respect to reduced left and right compositions. For the same reason the notions of adjunction and equivalence of sesqui-categories (w.r.t. strict transformations) are straightforward generalization of their categorical analogues.

Generalizing sesqui-categories (Chapter 3) we will need a further notion of sesqui-transformation, whose definition follows

Definition 4.6 (lax sesqui-transformations)**.** *Let two parallel sesqui-functors*

$$\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$$

be given, and let be given a 2-graph transformation $\Gamma : \mathcal{F} \Rightarrow \mathcal{G}$*. Then a lax natural transformation* $\Gamma : \mathcal{G} \Rightarrow \mathcal{G}$ *is given by the following data:*
• *For every object* A *of* $\mathcal{C}$*, an arrow*

$$\Gamma_A : \mathcal{F}(A) \to \mathcal{G}(A)$$

• (naturality w.r.t. 1-cells) *For every arrow* $f : A \to B$ *of* $\mathcal{C}$*, a 2-cell*

$$\Gamma_f : \Gamma_A \bullet \mathcal{G}(f) \Rightarrow \mathcal{F}(f) \bullet \Gamma_B$$

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\Gamma_A} & \mathcal{G}(A) \\ {\scriptstyle \mathcal{F}(f)}\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle \Gamma_f} & \downarrow{\scriptstyle \mathcal{G}(f)} \\ \mathcal{F}(B) & \xrightarrow[\Gamma_B]{} & \mathcal{G}(B) \end{array}$$

• (naturality w.r.t. 2-cells) *For every 2-cell* $\alpha : f \Rightarrow g : A \to B$ *in* $\mathcal{C}$*, an equation*

$$\begin{array}{ccc} \Gamma_A \bullet \mathcal{G}(f) & \xRightarrow{\Gamma_A \bullet_L \mathcal{G}(\alpha)} & \Gamma_A \bullet \mathcal{G}(g) \\ {\scriptstyle \Gamma_f}\Downarrow & & \Downarrow{\scriptstyle \Gamma_g} \\ \mathcal{F}(f) \bullet \Gamma_B & \xRightarrow[\mathcal{F}(\alpha)\bullet_R]{} & \mathcal{F}(g) \bullet \Gamma_B \end{array}$$

These data have to satisfy usual functoriality axioms, i.e. for every object A *of* $\mathcal{C}$ $\Gamma_{1_A} = 1_{\Gamma_A}$*, and for each pair of composeable arrows* f, h*,* $(\Gamma_f \bullet \mathcal{G}(h))(\mathcal{F}(f) \bullet \Gamma_g) = \Gamma_{fh}$

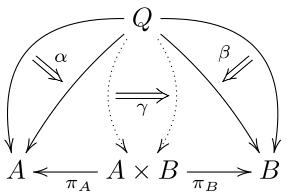
Let us notice that, in general, a lax sesqui-transformation is *not* a natural transformation of the functors underlying domain and co-domain sesqui-functors.

4.2 Finite products and h-pullbacks in a sesqui-category

In the sesqui-categorical context we will refer to binary products according to the following 2-dimensional universal property

Definition 4.7. *Let* $\mathcal{C}$ *be a sesqui-category,* A *and* B *two objects of* $\mathcal{C}$*. A product of* A *and* B *is a triple* $(A \times B, \pi_A : A \times B \to A, \pi_B : A \times B \to B)$ *satisfying the following universal property:*
for every object Q *of* $\mathcal{C}$ *and 2-cells* $\alpha : a \Rightarrow a' : Q \to A$ *,* $\beta : b \Rightarrow b' : Q \to B$ *,*
there exists a unique 2-cell $\gamma : q \Longrightarrow q' : Q \longrightarrow A \times B$ *with* $\gamma \bullet \pi_A = \alpha$ *and* $\gamma \bullet \pi_B = \beta$ *(write* $\gamma = \langle \alpha, \beta \rangle$*).*

The situation may be visualized on the diagram below



It is easy to show that such a product satisfies also the universal property defining categorical products.

Definition 4.8. *Let $\mathcal{C}$ be a sesqui-category. A terminal object is an object I of $\mathcal{C}$ satisfying the following universal property:*
for every other object X of $\mathcal{C}$, there exists a unique 2-cell $\xi : x \Rightarrow x' : X \to I$.

Products and terminals defined this way are determined up to isomorphism. Furthermore finite products and canonical isomorphisms are defined as in the categorical case.
Now we consider the 2-cells $\alpha : f \Rightarrow g : A \to B$ and $\beta : h \Rightarrow k : C \to D$ in a sesqui-category $\mathcal{C}$. A 2-cell

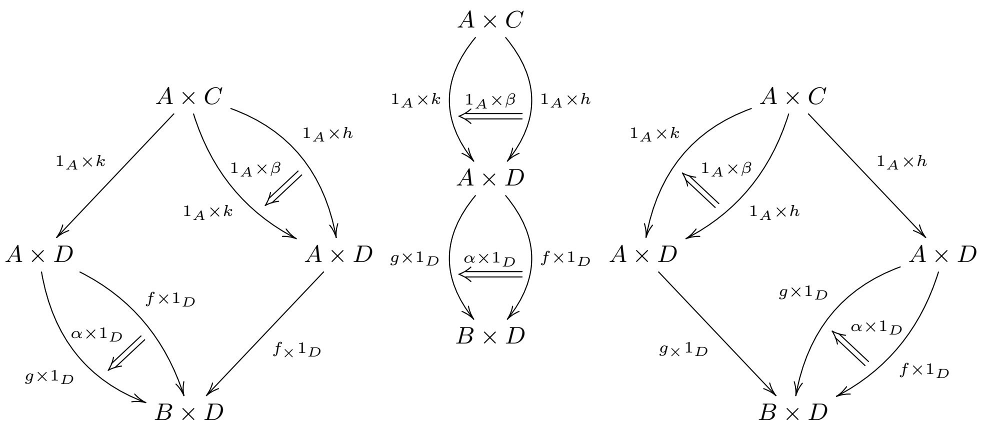
$$\alpha \times \beta : f \times h \Rightarrow g \times k : A \times C \to B \times D$$

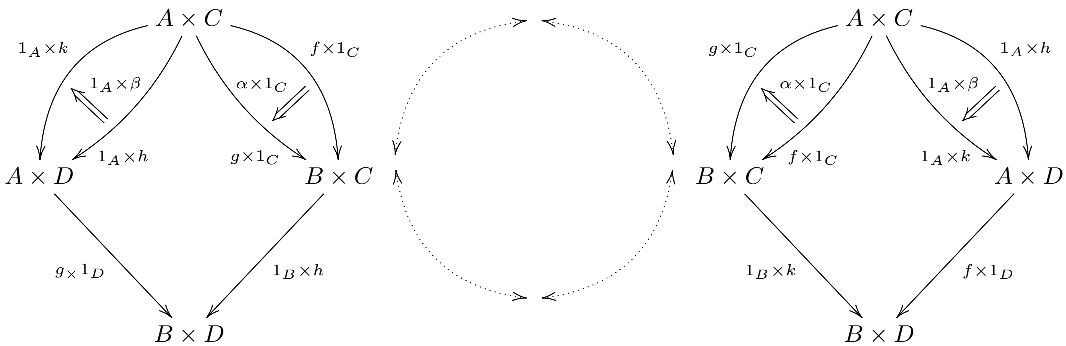
is uniquely determined by the universal property: $\alpha \times \beta = \langle \pi_A \bullet \alpha, \pi_C \bullet \beta \rangle$.
Notice that this induces a kind of commutative horizontal composition of 2-cells, provided they are on different product-components.
In fact, we need the following

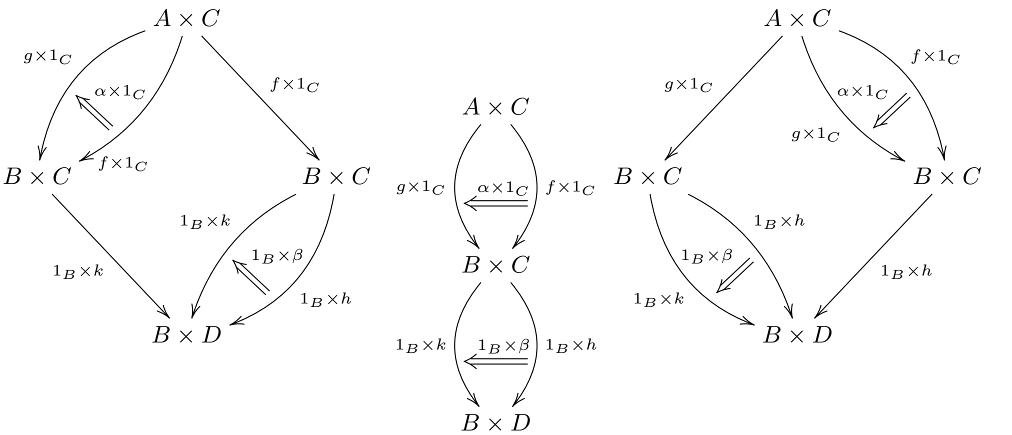
Lemma 4.9. *For α and β as above,*

$$((1_A{\times}\beta)\bullet(f{\times}1_D))\cdot((1_A{\times}k)\bullet(\alpha{\times}1_D)) = ((1_A{\times}h)\bullet(\alpha{\times}1_D))\cdot((1_A{\times}\beta)\bullet(g{\times}1_D))$$

Refer to [Met08b] for a proof.







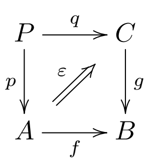
Lemma 4.9 allows us to define a horizontal composition of this kind of 2-cells

$$(1_A \times \beta) \bullet (\alpha \times 1_D) = \alpha \times \beta = (\alpha \times 1_C) \bullet (1_B \times \beta)$$

and to prove diagram equalities, such as the one above. These kind of diagrammatic equations will be called *product interchange rules.*

We introduce here a notion of standard h-pullback suitable for our purposes. This notion has been formalized by Michael Mather in [Mat76b], for generic categories of spaces, with (eventually pointed) topological spaces in mind. It has been further generalized to h-categories[2] by Marco Grandis in [Gra94]. We (ab)use the term h-pullback, instead of that of *comma-square* because our first aim is to develop a theory in the context of n-groupoids, [KMV08b] or at least in the contest of n-categories with respect to equivalence n-transformations, where the two notions coincide.

Definition 4.10. *Consider two arrows $f : A \to B$ $g : C \to B$ in a sesqui-category $\mathcal{C}$. An h-pullback of f and g is a four-tuple $(P(f,g), p, q, \varepsilon)$*



[2]A h-category is a weaker notion than that of a sesqui-category, see [Gra94].

such that for any other four-tuple $(X, m, n, \lambda mf \Rightarrow ng)$ *there exists a unique* $\ell : X \to P$ *satisfying* $\ell p = m$, $\ell q = n$, $\ell \bullet_L \varepsilon = \lambda$.

In [Met08b] one can find a proof that Definition above defines pullbacks up to isomorphisms, and that of the following

Lemma 4.11 (Pullback of h-projections.)**.** *In the sesqui-category* $\mathcal{C}$*, let be given the diagram below, where the left-hand square is commutative and the right-hand square* ε *is a* h*-pullback*

$$\begin{array}{ccccc} R & \xrightarrow{s} & P & \xrightarrow{q} & D \\ {\scriptstyle r}\downarrow & & {\scriptstyle p}\downarrow & \overset{\varepsilon}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[e]{} & B & \xrightarrow[f]{} & C \end{array}$$

then the composition $s \bullet_L \varepsilon$ *is a* h*-pullback if, and only if, the left hand square is a pullback.*

4.3 Sesqui^2-categories

The necessity of introducing 3-morphisms (lax-n-modifications) takes us out of the comfortable setting of sesqui-categories, into the unknown territory of sesqui-categorically enriched structures.
Following this suggestion, we have named the new setting *sesqui*^{2}*-category.* This notion is closely related with that of Tas (Tas, pl. Teisi, are mathematical objects introduced by the pioneering work of S. Crans, see [Cra00, Cra01]) and incorporates a horizontal dimension raising horizontal composition of 2-morphisms. A special example of sesqui^2-category is given by the well-known notion of *Gray*-category [Gra76, Gra74]. There, horizontal composition of 2-morphisms is always an identity 3-morphism, therefore homs are indeed very special sesqui-categories, i.e. 2-categories, and those identiy 3-morphisms imply interchange law for horizontal compositions.
Now, *Gray*-categories are indeed enriched in 2-Cat, hence, in order to fully justify the name *sesqui*^{2}*-category* , it would be interesting to investigate explicitly the enrichment that generates this notion from that of sesqui-category [Met08a]. We leave this issue to further investigations.
What we present here is a treatable inductive approach, comprehensive of a useful characterization given in *Theorem 4.14*.

Definition 4.12. *A (small)* sesqui^2-category $\mathcal{C}$ *consists of:*

- *A 3-truncated reflexive globular set* $\mathcal{C}_\bullet$*:*

$$\mathcal{C}_3 \;\underset{c_2}{\overset{d_2}{\rightrightarrows}}\!\!\overset{e_2}{\leftarrow}\; \mathcal{C}_2 \;\underset{c_1}{\overset{d_1}{\rightrightarrows}}\!\!\overset{e_1}{\leftarrow}\; \mathcal{C}_1 \;\underset{c_0}{\overset{d_0}{\rightrightarrows}}\!\!\overset{e_0}{\leftarrow}\; \mathcal{C}_0$$

with operations

$$\bullet^m : \ \mathcal{C}_p \ {}_{c_m}\!\!\times_{d_m} \mathcal{C}_q \to \mathcal{C}_{p+q-m-1}, \qquad m < \min(p, q)$$

such that the following axioms hold:

(i) *For every pair* $\mathbb{C}, \mathbb{D} \in \mathcal{C}_0$*, the localization* $\mathcal{C}(\mathbb{C}, \mathbb{D})$ *is a sesqui-category, with*

- *object are* $F, G, etc. \in \mathcal{C}_1(\mathbb{C}, \mathbb{D})$
- *for any pair of objects* F, G*, 1-cells are* $\alpha, \beta, etc. \in \mathcal{C}_2(F, G)$
- *for any pair of 1-cells* $\alpha, \gamma : F \to G$*, 2-cells are* $\Lambda, \Sigma, etc. \in \mathcal{C}_3(\alpha, \beta)$

k-compositions are restrictions of $\bullet^{k+1}$*-compositions:*

- *0-composition of 1-cells of* $\mathcal{C}(\mathbb{C}, \mathbb{D})$

$$\diamond^0 := \bullet^1 : \mathcal{C}_2 \ {}_{c_1}\!\times_{d_1} \mathcal{C}_2 \to \mathcal{C}_2$$

- *left/right reduced 0-compositions of 1-cell with a 2-cell of* $\mathcal{C}(\mathbb{C}, \mathbb{D})$

$$\diamond^0_L := \bullet^1 : \mathcal{C}_2 \ {}_{c_1}\!\times_{d_1} \mathcal{C}_3 \to \mathcal{C}_3$$

$$\diamond^0_R := \bullet^1 : \mathcal{C}_3 \ {}_{c_1}\!\times_{d_1} \mathcal{C}_2 \to \mathcal{C}_3$$

- *1-compositions of 2-cells of* $\mathcal{C}(\mathbb{C}, \mathbb{D})$

$$\diamond^1 := \bullet^2 : \mathcal{C}_3 \ {}_{c_2}\!\times_{d_2} \mathcal{C}_3 \to \mathcal{C}_3$$

(ii) *For every morphism* $F : \mathbb{C} \to \mathbb{D}$ *and objects* $\mathbb{B}, \mathbb{E}$ *of* $\mathcal{C}$

$$- \bullet^0 F : \ \mathcal{C}(\mathbb{B}, \mathbb{C}) \to \mathcal{C}(\mathbb{B}, \mathbb{D})$$

$$F \bullet^0 - : \ \mathcal{C}(\mathbb{D}, \mathbb{E}) \to \mathcal{C}(\mathbb{C}, \mathbb{E})$$

are sesqui-functors.
(iii) *For every object* $\mathbb{C}$ *and objects* $\mathbb{B}, \mathbb{D}$ *of* $\mathcal{C}$*, if we denote* $id_{\mathbb{C}} = e_0(\mathbb{C})$*,*

$$- \bullet^0 id_{\mathbb{C}} : \ \mathcal{C}(\mathbb{B}, \mathbb{C}) \to \mathcal{C}(\mathbb{B}, \mathbb{C})$$

$$id_{\mathbb{C}} \bullet^0 - : \ \mathcal{C}(\mathbb{C}, \mathbb{D}) \to \mathcal{C}(\mathbb{C}, \mathbb{D})$$

are identity sesqui-functors.
(iv) (naturality axioms) *For every pair of 0-composable 2-morphisms* $\alpha : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$ *and* $\beta : H \Rightarrow K : \mathbb{D} \to \mathbb{E}$

$$(a) \quad \alpha \bullet^0 \beta : (F \bullet^0 \beta) \bullet^1 (\alpha \bullet^0 K) \to (\alpha \bullet_0 H) \bullet^1 (G \bullet^0 \beta)$$

For every 2-morphisms $\varepsilon : L \Rightarrow M : \mathbb{B} \to \mathbb{C}$ *and* $\beta : H \Rightarrow K : \mathbb{D} \to \mathbb{E}$*, and for every 3-morphism* $\Lambda : \alpha \implies \omega : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$

$$(b) \quad (\alpha \bullet^0 \beta) \bullet^2 \Big((\Lambda \bullet^0 H) \bullet^1 (G \bullet^0 \beta) \Big) = \Big((F \bullet^0 \beta) \bullet^1 (\Lambda \bullet^0 K) \Big) \bullet^2 (\omega \bullet^0 \beta)$$

$$(c) \quad \Big((L \bullet^0 \Lambda) \bullet^1 (\varepsilon \bullet^0 G) \Big) \bullet^2 (\varepsilon \bullet^0 \omega) = (\varepsilon \bullet^0 \alpha) \bullet^2 \Big((\varepsilon \bullet^0 F) \bullet^1 (M \bullet^0 \Lambda) \Big)$$

(v) (functoriality axioms) *For every 2-morphisms* $\omega : D \Rightarrow E : \mathbb{B} \to \mathbb{C}$ *and* $\gamma : H \Rightarrow L : \mathbb{D} \to \mathbb{E}$ *and every pair of 1-composable 2-morphisms* $\alpha : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$ *and* $\beta : G \Rightarrow H : \mathbb{C} \to \mathbb{D}$

$$(a) \quad (\alpha \bullet^1 \beta) \bullet^0 \gamma = \Big((\alpha \bullet^0 \gamma) \bullet^1 (\beta \bullet^0 L) \Big) \bullet^2 \Big((\alpha \bullet^0 K) \bullet^1 (\beta \bullet^0 \gamma) \Big)$$

(b) $\quad \omega \bullet^0 (\alpha \bullet^1 \beta) = \Big((\omega \bullet^0 \alpha) \bullet^1 (E \bullet^0 \beta)\Big) \bullet^2 \Big((D \bullet^0 \alpha) \bullet^1 (\omega \bullet^0 \beta)\Big)$

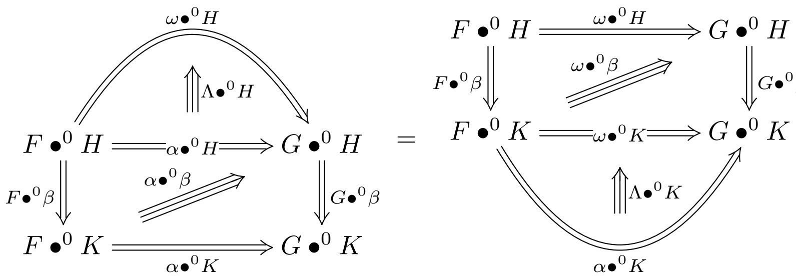
(vi) (associativity axiom) *For every 0-composable triple* $x \in [\mathcal{C}(\mathbb{B},\mathbb{C})]_p$, $y \in [\mathcal{C}(\mathbb{C},\mathbb{D})]_q$ *and* $z \in [\mathcal{C}(\mathbb{D},\mathbb{E})]_r$, *with* $p+q+r \leq 2$

$$(x \bullet^0 y) \bullet^0 z = x \bullet^0 (y \bullet^0 z)$$

(vii) (identity axioms) *For morphisms* $E : \mathbb{B} \to \mathbb{C}$ *and* $H : \mathbb{D} \to \mathbb{E}$, *and 2-morphism* $\alpha : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$,

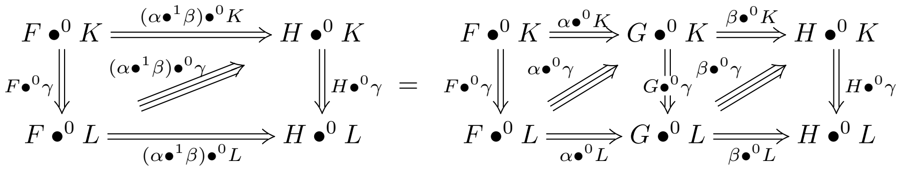
$$id_F \bullet^0 \alpha = id_{F \bullet^0 \alpha}, \qquad \alpha \bullet^0 id_G = id_{\alpha \bullet^0 G}$$

Remark 4.13. 1. Axiom $(iv)(c)$ is better understood when visualized as in the following diagram (same notation)



The same can be claimed for axiom $(iv)(b)$.

2. Axiom $(v)(a)$ is better understood when visualized as in the following diagram (same notation)



The same can be claimed for axiom $(v)(b)$.

Theorem 4.14. *Let* $\mathcal{C}_\bullet$ *be a 3-truncated reflexive globular set. Then the following two statements are equivalent.*

1. $\mathcal{C}$ *is a (small)* sesqui^2-category
2. *Axioms* (i), (ii) *and* (iii) *of* Definition 4.12 *hold, moreover*

$(viii)$ *The 2-truncation* $\mathcal{C}_2 \underset{c_1}{\overset{d_1}{\substack{\longrightarrow \\ \xleftarrow{e_1} \\ \longrightarrow}}} \mathcal{C}_1 \underset{c_0}{\overset{d_0}{\substack{\longrightarrow \\ \xleftarrow{e_0} \\ \longrightarrow}}} \mathcal{C}_0$ *of* $\mathcal{C}_\bullet$ *is a sesqui-category.*

(ix) *For every 2-morphism* $\alpha : F \Rightarrow G : \mathbb{C} \to \mathbb{D}$ *and objects* $\mathbb{B}, \mathbb{E}$ *of* $\mathcal{C}$

$$- \bullet^0 \alpha : \; - \bullet^0 F \Rightarrow - \bullet^0 G : \; \mathcal{C}(\mathbb{B},\mathbb{C}) \to \mathcal{C}(\mathbb{B},\mathbb{D})$$

$$\alpha \bullet^0 - : \; F \bullet^0 - \Rightarrow G \bullet^0 - : \; \mathcal{C}(\mathbb{D},\mathbb{E}) \to \mathcal{C}(\mathbb{C},\mathbb{E})$$

are lax *natural transformations of sesqui-functors.*

(x) *For every morphism* $F: \mathbb{C} \to \mathbb{D}$ *of* $\mathcal{C}$

$$- \bullet^0 id_F \ \ - \bullet^0 F \Rightarrow - \bullet^0 F$$

$$id_F \bullet^0 - : \ F \bullet^0 - \Rightarrow F \bullet^0 -$$

are identical natural transformations.

(xi) (reduced associativity axiom)

For every 0-composable triple $x \in [\mathcal{C}(\mathbb{B}, \mathbb{C})]_p$, $y \in [\mathcal{C}(\mathbb{C}, \mathbb{D})]_q$ *and* $z \in [\mathcal{C}(\mathbb{D}, \mathbb{E})]_r$, *with* $p + q + r = 2$

$$(x \bullet^0 y) \bullet^0 z = x \bullet^0 (y \bullet^0 z)$$

i.e. for 3-morphism Λ, *2-morphisms* α, β *and morphisms* F, G *of* $\mathcal{C}$, *the following equations hold, when composites exist:*

$$\begin{array}{ll} (\Lambda \bullet^0 F) \bullet^0 G = \Lambda \bullet^0 (F \bullet^0 G) & (\alpha \bullet^0 \beta) \bullet^0 F = \alpha \bullet^0 (\beta \bullet^0 F) \\ (F \bullet^0 \Lambda) \bullet^0 F = F \bullet^0 (\Lambda \bullet^0 G) & (\alpha \bullet^0 F) \bullet^0 \beta = \alpha \bullet^0 (F \bullet^0 \beta) \\ (\Lambda \bullet^0 F) \bullet^0 G = \Lambda \bullet^0 (F \bullet^0 G) & (F \bullet^0 \alpha) \bullet^0 \beta = F \bullet^0 (\alpha \bullet^0 \beta) \end{array}$$

Proof. First we prove that 1. implies 2..
Condition $(viii)$ is equivalent to satisfying properties $(L1)$ to $(L4)$, $(R1)$ to $(R4)$ and $(LR5)$ of *Proposition 4.2.* Now, $(L1)$ and $(R1)$ hold by (iii), $(L2)$ and $(R2)$ by (iv), $(L3)$, $(R3)$, $(L4)$ and $(R4)$ by (ii), $(LR5)$ by (vi).
Condition (ix) holds. In fact let us recall *Definition 4.6.* Assignment on objects (=1-cells) is given by 0-composition, naturality by (iv) and functoriality by (v) (compositions) and (vii) (units).
Condition (x) holds too. In fact this is implied by (ix) above and (vii).
Finally (xi) is a subset of (vi).

Conversely we prove that 2. implies 1..
Conditions (iv) and (v) hold by (ix).
Condition (vi) holds by (xi) for the cases $p + q + r = 2$. What is still to prove is the case $p + q + r = 0$ and the case $p + q + r = 1$, that are given by $(viii)$.
Finally (vii) is a consequence of (ix) and (x). □

Remark 4.15. Notice that the characterization given by *Theorem 4.14* is somehow redundant. Nevertheless its usefulness is that it makes available practical rules in order to deal with calculations in a sesqui^2-categorical environment.

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