Braided and symmetric internal groupoids

Giuseppe Metere

Università di Milano
Milano, Italy

CatAlg 2011
Gargnano del Garda (BS)

Overview

- Intro
- Crossed Modules and Weak Morphisms
- Braiding and Symmetry

Intro

We are concerned with the study of the algebraic properties of categories internal to a semi-abelian category $\mathcal{E}$ (leading example $\mathcal{E} = \mathbf{Gp}$, the category of groups).

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "m"] & C_1 \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "d"'] & C_0 \arrow[l, "e" description] \end{tikzcd}$$

Since $\mathcal{E}$ Mal'cev, $\mathbf{Cat}(\mathcal{E}) = \mathbf{Gpd}(\mathcal{E})$.

Fact. A category in **Gp** is the same as a group in **Cat**, i.e. a strict 2-group.

The weak version of this, **categorical groups**, arose in algebraic geometry (gr-categories). They have been extensively studied and notions such as that of commutativity, of exact sequences, factorization system, etc. have been introduced and studied.

Weak morphisms

Is it possible to develop a similar theory in an intrinsic setting?

To answer this question, it is important to decide how the objects organize in a 2-category.

For the 2-category of 2-groups (strict categorical groups) there are (at least) two meaningful notions of morphisms:

- internal functors
- monoidal functors

Weak morphisms

Internal functors... ($\Leftrightarrow$ Strict monoidal functors)

$$\begin{tikzcd} H_1 \arrow[r, "F_1"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & G_1 \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\ H_0 \arrow[u] \arrow[r, "F_0"'] & G_0 \arrow[u] \end{tikzcd}$$

i.e. (F_1, F_0) are group homomorphisms,
compatible with the categorical structure of $\mathbb{H}$ and $\mathbb{G}$.
The corresponding 2-category is denoted $\mathbf{2Gp}_{\mathrm{str}}$.

Weak morphisms

Monoidal functors...

$$\begin{tikzcd} H_1 \arrow[r, dashed, "F_1"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & G_1 \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\ H_0 \arrow[u] \arrow[r, dashed, "F_0"'] & G_0 \arrow[u] \end{tikzcd}$$

i.e. (F_1, F_0) are functions in **Set**,
compatible with the categorical structure of $\mathbb{H}$ and $\mathbb{G}$, and with the group operations **only up to isomorphisms**.
The corresponding 2-category is denoted **2Gp**.

Weak morphisms

Many important properties of 2-groups cannot be observed with only the strict monoidal functors available.
Need an internal notion of (weak) monoidal functor.

Theorem (Vitale 2010) The embedding $\mathbf{2Gp}_{\mathrm{str}} \to \mathbf{2Gp}$ is the bicategory of fractions of $\mathbf{2Gp}_{\mathrm{str}}$, w.r.t. the class of internal weak equivalences.

The analogous result holds for $\mathbf{2Lie} = \mathbf{Gpd}(\mathbf{Lie})$

Weak morphisms

Theorem (Mantovani, M., Vitale 2011) Let $\mathcal{E}$ be Barr-exact. The bicategory of fractions of $\mathbf{Gpd}(\mathcal{E})$ w.r.t. (internal) weak equivalences can be described by **fractors**, i.e. profunctors

$$\mathbb{H} \overset{E}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} \mathbb{G}$$

whose canonical span representation has the left leg a surjective weak equivalence.

Fractors organize in a bicategory $\mathbf{Fract}(\mathcal{E})$.

Fractors give a notion of weak morphism of internal groupoids in $\mathcal{E}$, equivalent to that of monoidal functors when $\mathcal{E} = \mathbf{Gp}$.

Crossed modules in $\mathcal{E}$ - I

For making computations easier with 2-groups and strict monoidal functors, one can use **crossed modules of groups**.

This notion has been internalized in the semi-abelian context by G. Janelidze, so that **internal crossed modules** can be used for computing with internal groupoids and internal functors.

Crossed modules in $\mathcal{E}$ - II

An internal crossed module $\mathbb{G}$ in a semi-abelian category $\mathcal{E}$ [J 2003], with "Smith = Huq" can be described [MF VdL 2010] as a pair

$$G_0 \flat G \xrightarrow{\xi} G \xrightarrow{\partial} G_0$$

making the diagrams commute:

$$\begin{tikzcd}
G \flat G \arrow[r, "\chi_G"] \arrow[d, "\partial \flat 1"'] & G \arrow[d, "1"] \\
G_0 \flat G \arrow[r, "\xi"] \arrow[d, "1 \flat \partial"'] & G \arrow[d, "\partial"] \\
G_0 \flat G_0 \arrow[r, "\chi_{G_0}"'] & G_0
\end{tikzcd}$$

A (strict) morphism of crossed modules $\mathbb{H} \longrightarrow \mathbb{G}$ is a pair of equivariant morphisms $F \colon H \longrightarrow G$, $F_0 \colon H_0 \longrightarrow G_0$.

Crossed modules in $\mathcal{E}$ - III

Theorem. (Janelidze 2010) There is an equivalence of categories

$$\underline{\mathbf{Xmod}}(\mathcal{E}) \simeq \underline{\mathbf{Gpd}}(\mathcal{E})$$

Exercise. The equivalence above underlies a bi-equivalence of 2-categories

$$\mathbf{Xmod}(\mathcal{E}) \simeq \mathbf{Gpd}(\mathcal{E})$$

Crossed modules in $\mathcal{E}$ - IV

We can extend the biequivalence:

$$\begin{array}{ccc}
\mathbf{Xmod}(\mathcal{E}) & \simeq & \mathbf{Gpd}(\mathcal{E}) \\
\downarrow & & \downarrow \\
?\mathbf{Butt}(\mathcal{E}) & \simeq & \mathbf{Fract}(\mathcal{E})
\end{array}$$

As fractors model weak morphisms of groupoids, there is a notion of weak morphism of crossed modules, that corresponds to fractors under the (bi)equivalence: **butterflies**.

Internal butterflies - I

Butterflies were introduced by B. Noohi in [Noo05] for the category of groups. Here we recall their internal definition [AMMV11]. A butterfly $E\colon \mathbb{H} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{G}$:

$$
\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
H_0 & & G_0
\end{tikzcd}
\qquad
\begin{tikzcd}
E\flat H \arrow[r, "\sigma\flat 1"] \arrow[d, "1\flat\kappa"'] & H_0\flat H \arrow[r, "\xi"] & H \arrow[d, "\kappa"] \\
E\flat E \arrow[rr, "\chi_E"'] & & E
\end{tikzcd}
$$

$$
\begin{tikzcd}
E\flat G \arrow[r, "\rho\flat 1"] \arrow[d, "1\flat\iota"'] & G_0\flat G \arrow[r, "\xi"] & G \arrow[d, "\iota"] \\
E\flat E \arrow[rr, "\chi_E"'] & & E
\end{tikzcd}
$$

i. (κ, ρ) is a complex
ii. (ι, σ) is an extension
iii. iv. the two diagrams on the right commute

Internal Butterflies - II

Fact: butterflies correspond to fractors.

$$
\begin{tikzcd}
 & H \times G \arrow[dl, "\pi_H"'] \arrow[dd, "\kappa\sharp\iota"] \arrow[dr, "\pi_G"] & \\
H \arrow[dr, "\kappa"] \arrow[dd, "\partial"'] & & G \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \\
 & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
H_0 & & G_0
\end{tikzcd}
$$

Composition, identities and 2-morphisms of butterflies can be obtained from the corresponding notions for fractors.

Internal Butterflies - III

A 2-cells $E \Rightarrow E'$ corresponds to a morphism in $\mathcal{E}$ $f : E \to E'$ s.t. all the following diagrams commute:

$$\begin{tikzcd}
H \arrow[ddd, "\partial"'] \arrow[dr, "\kappa"] \arrow[ddr, "\kappa'"'] & & G \arrow[ddd, "\partial"] \arrow[dl, "\iota"'] \arrow[ddl, "\iota'"] \\
& E \arrow[d, "f"] \arrow[ddl, "\delta"] \arrow[ddr, "\gamma"'] & \\
& E' \arrow[dl, "\delta'"] \arrow[dr, "\gamma'"'] & \\
H_0 & & G_0
\end{tikzcd}$$

Butterflies and 2-cells form a locally groupoidal bicategory $\mathbf{Butt}(\mathcal{E})$. Examples

Kernels of butterflies - I

We can use butterflies in order to apply the methods used for 2-groups in a wider context.

Example: The kernel of a Butterfly

We translate the construction of the standard h-kernel for 2-groups:

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "K"'] \arrow[rr, bend left, "0"] & \mathbb{H} \arrow[r, "E"'] \arrow[u, phantom, "e\Uparrow"] & \mathbb{G}
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[rr, "1"] \arrow[dd, dashed, "\partial"'] & & H \arrow[dr, "\kappa"] \arrow[dd, "\partial"'] & & G \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \\
& & & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
K_0 \arrow[r, "\ker\rho"'] & E \arrow[r, "\sigma"'] & H_0 & & G_0
\end{tikzcd}$$

We obtain a crossed module $\mathbb{K}$, a morphism $K\colon \mathbb{K} \to \mathbb{H}$ and a 2-morphism $NE \Rightarrow 0$ universal w.r.t. (homotopic) universal property.

Kernels of butterflies - II

Recall from [E K VdL 2005] that the (only non-trivial) **homology objects** of a crossed module $\partial\colon H \to H_0$ are

$$\begin{aligned} \mathcal{H}_0(\partial\colon H \to H_0) &= \mathrm{coker}(\partial) \\ \mathcal{H}_1(\partial\colon H \to H_0) &= \ker(\partial) \end{aligned}$$

They correspond to the homotopy invariants π_0 (connected components) and π_1 (automorphism of 0), so that weak equivalences coincide with homology isomorphisms.

This fact has applications. . .

Kernels of butterflies - III

Proposition: From a kernel diagram

$$\begin{tikzcd} \mathbb{K} \arrow[r, "K"'] \arrow[rr, bend left, "0"] & \mathbb{H} \arrow[u, phantom, "e\Uparrow"] \arrow[r, "E"', "+" marking] & \mathbb{G} \end{tikzcd}$$

one can get the long exact sequence:

$$0 \to \mathcal{H}_1(\mathbb{K}) \to \mathcal{H}_1(\mathbb{H}) \to \mathcal{H}_1(\mathbb{G}) \xrightarrow{\delta} \mathcal{H}_0(\mathbb{K}) \to \mathcal{H}_0(\mathbb{H}) \to \mathcal{H}_0(\mathbb{G})$$

▸ Proof.

Cokernels of butterflies

Can we construct cokernels of butterflies as we have done for kernels?

$$
\begin{tikzcd}
H \arrow[d, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dl, "\iota"'] \arrow[d, "\partial"] \arrow[r, "\iota"] & E \arrow[r, "\operatorname{cok}\kappa"] & C \arrow[d, dashed, "\partial"] \\
 & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & & & \\
H_0 & & G_0 \arrow[rr, "1"'] & & G_0
\end{tikzcd}
$$

This way we do not obtain a crossed module: the arrow $\partial\colon C \to G_0$ is just a morphism in $\mathcal{E}$.

Again, the case of 2-groups shows the way: $E\colon \mathbb{H} \mathrel{\longrightarrow\!\!\!\!\!\!+\;} \mathbb{G}$ needs to be **braided**.

Braiding

Braidings and symmetries are higher dimensional generalizations of the notion of **the commutativity of an algebraic operation**.

At the (1-)categorical level this condition is internal: for $\mathcal{E}$ unital, an object G is **commutative** if it is endowed with a magma structure, i.e. there exists an "operation"

$$P\colon G \times G \longrightarrow G$$

that makes the triangles commute

$$\begin{tikzcd} G \arrow[r, "\langle 1,0 \rangle"] \arrow[dr, "1"'] & G \times G \arrow[d, "P"] & G \arrow[l, "\langle 0,1 \rangle"'] \arrow[dl, "1"] \\ & G & \end{tikzcd}$$

This condition is too strong if applied to internal groupoids or to crossed modules.

Braiding

A notion of braided crossed module (of groups) comes from homotopy theory – from the Samelson product [W 1974].

For the case of 2-groups, the notion of braiding has been developed in the wider context of monoidal categories by Joyal and Street [J S 1986].

We start from the last, since 2-groups better fit the conceptual framework: we will come back to crossed modules *via* butterflies.

Braided 2-groups - I

A braided 2-group [A. Joyal R. Street 1986] is a 2-group $(\mathbb{G}, +, 0)$ equipped with a braiding function $t\colon G_0 \times G_0 \dashrightarrow G_1$ such that:

1. $t(x,y)\colon x+y \xrightarrow{\sim} y+x$
2. $$\begin{tikzcd} x+y \arrow[r, "f+g"] \arrow[d, "t(x,y)"'] & x'+y' \arrow[d, "t(x',y')"] \\ y+x \arrow[r, "g+f"'] & y'+x' \end{tikzcd}$$
3. $$\begin{tikzcd} x+y+z \arrow[r, "t+1"] \arrow[dr, "t"'] & y+x+z \arrow[d, "1+t"] \\ & y+z+x \end{tikzcd} \qquad \begin{tikzcd} x+y+z \arrow[r, "1+t"] \arrow[dr, "t"'] & x+z+y \arrow[d, "t+1"] \\ & z+x+y \end{tikzcd}$$

The braiding is symmetric if moreover

4. $t(y,x) = t(x,y)^{-1}$

Braided 2-groups - II

The following facts are equivalent:

- $\mathbb{G}$ is braided
- $+\colon\ \mathbb{G}\times\mathbb{G} \longrightarrow \mathbb{G}$ is monoidal
- $\mathbb{G}$ is a **weak commutative object** in **2Gp**, i.e. there exist P monoidal and two 2-iso ℓ and r

$$\begin{tikzcd}
\mathbb{G} \arrow[r, "J_1"] \arrow[dr, "1"'] & \mathbb{G}\times\mathbb{G} \arrow[d, "P"] \arrow[dl, phantom, "\Leftarrow\ell"] \arrow[dr, phantom, "r\Rightarrow"] & \mathbb{G} \arrow[l, "J_2"'] \arrow[dl, "1"] \\
& \mathbb{G} &
\end{tikzcd}$$

Only the third notion is internal. . .

Braided internal groupoids

Definition.

A **braided internal groupoid** is a $\mathbb{G}$ equipped with a fractor $P\colon \mathbb{G}\times\mathbb{G} \longrightarrow \mathbb{G}$ and two 2-isomorphisms that make it a weak commutative object in $\mathbf{Fract}(\mathcal{E})$.

A **morphism of braided groupoids** is a fractor between them compatible with the braidings up to 2-isomorphism.

Now we can see how we can rid of the 2-cells in the definition and give a description of braided crossed modules.

Braided internal crossed modules - I

Definition - Proposition. A crossed module $\mathbb{G}$ is braided (give rise to a braided groupoid) if(f) it is equipped with a butterfly P and two morphisms s_1, s_2

$$\begin{tikzcd} G \times G \arrow[dd, "\partial\times\partial"'] \arrow[dr, "\alpha"] & & G \arrow[dl, "\beta"'] \arrow[dd, "\partial"] \\ & P \arrow[dl, "\gamma"] \arrow[dr, "\delta"'] & \\ G_0 \times G_0 & & G_0 \end{tikzcd} \qquad \begin{tikzcd} G_0 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "s_2"'] & P \end{tikzcd}$$

such that the diagrams commute:

$$\begin{tikzcd} G_0 \arrow[r, "s_i"] \arrow[dr, "j_i"'] & P \arrow[d, "\gamma"] \\ & G_0 \times G_0 \end{tikzcd} \qquad \begin{tikzcd} G_0 \arrow[r, "s_i"] \arrow[dr, "1"'] & P \arrow[d, "\delta"] \\ & G_0 \end{tikzcd} \qquad \begin{tikzcd} G_0 \arrow[r, "\langle j_i, 1\rangle"] \arrow[d, "\partial"'] & G \times G \times G \arrow[d, "\alpha\sharp\beta"] \\ G_0 \arrow[r, "s_i"'] & P \end{tikzcd}$$

Braided internal crossed modules - II

Remark. Given a braiding (P, s_1, s_2) on a crossed module $\mathbb{G}$ we define the morphism $c_G \colon (G_0|G_0) \to G$, that is the unique arrow that makes the diagram commute (... and a pullback):

$$\begin{tikzcd}
(G_0|G_0) \arrow[r, dashed, "c_G"] \arrow[d] & G \arrow[d, "\beta"] \\
G_0 + G_0 \arrow[r, "{[s_2, s_1]}"'] & P
\end{tikzcd}$$

This gives a connection with the classic notion of braided crossed module of groups.

Braided internal crossed modules - III

Definition. [??, Conduche 1983] A **braided crossed module** is a crossed module $G \xrightarrow{\partial} G_0$ endowed with a map

$$\{\ \ ,\ \ \} \colon G_0 \times G_0 \dashrightarrow G$$

such that, for any x, y, z in G_0, and a, b in G,

1. $\{x, y + z\} = y \cdot \{x, z\} + \{x, y\}$
2. $\{x + y, z\} = \{x, z\} + x \cdot \{y, z\}$
3. $\partial\{x, y\} = [y, x]$
4. $\{\partial a, x\} = x \cdot a - a$
5. $\{y, \partial b\} = b - y \cdot b$

Braided internal crossed modules - V

Internally, we do not have (yet!) a characterization of braided crossed modules in terms of the morphisms

$$c_G \colon (G_0|G_0) \longrightarrow G \ ,$$

but they seem to be relevant for some constructions, for instance, for the cokernel of a butterfly.

Braided internal crossed modules - IV

In order to understand the definition of braided crossed module we observe that the diagrams

$$\begin{tikzcd} G_0 \arrow[r, "\langle j_1,1\rangle"] \arrow[d, "\partial"'] & G\times G\times G \arrow[d, "\alpha\sharp\beta"] \\ G_0 \arrow[r, "s_1"'] & P \end{tikzcd} \qquad \begin{tikzcd} G_0 \arrow[r, "\langle j_2,1\rangle"] \arrow[d, "\partial"'] & G\times G\times G \arrow[d, "\alpha\sharp\beta"] \\ G_0 \arrow[r, "s_2"'] & P \end{tikzcd}$$

of the definition underlie two (strict) morphisms of crossed modules

$$S_1 : \mathbb{G} \longrightarrow \mathbb{P} \qquad S_2 : \mathbb{G} \longrightarrow \mathbb{P}$$

Braided internal groupoids - Reprise

We obtain the following characterization:

Proposition. A groupoid (crossed module) $\mathbb{G}$ is braided iff it is endowed with a fractor (butterfly)

$$\mathbb{G} \times \mathbb{G} \xrightarrow{P} \mathbb{G}$$

with canonical span representation $(\Gamma, \mathbb{P}, \Delta)$ and two internal functors (morphisms) $\mathbb{G} \xrightarrow{S_i} \mathbb{P}$, $i = 1, 2$, such that

$$S_i \Gamma = J_i$$

$$S_i \Delta = 1_{\mathbb{G}}$$

$$\begin{tikzcd}
 & \mathbb{G} \arrow[d, "S_i"] \arrow[ddl, bend right, "J_i"'] \arrow[ddr, bend left, "\mathrm{id}"] & \\
 & \mathbb{P} \arrow[dl, "\Gamma"] \arrow[dr, "\Delta"'] & \\
\mathbb{G} \times \mathbb{G} \arrow[rr, "P"'] & & \mathbb{G}
\end{tikzcd}$$

Symmetric internal groupoids I

The symmetry condition $t(y,x) = t(x,y)^{-1}$ can be re-stated by saying that t *is not only natural, but also monoidal,* or equivalently, in terms of the operation P.

This gives the definition of **symmetric internal groupoid**: a braided internal groupoid $(\mathbb{G}, P, S_1, S_2)$ with a 2-cell t

$$
\begin{tikzcd}
\mathbb{G}\times\mathbb{G} \arrow[dd, "\mathrm{Tw}"'] \arrow[dr, "P", "|" marking] & \\
{} \arrow[r, phantom, "\Downarrow t"] & \mathbb{G} \\
\mathbb{G}\times\mathbb{G} \arrow[ur, "P"', "|" marking] &
\end{tikzcd}
$$

Remark: also in the internal case, being symmetric does not add structure to the braiding: the 2-cell t, if it exists, is **unique**.

Symmetric internal groupoids II

It is remarkable to observe that **symmetry** may coincide with **braiding**:

Example: 2-Lie Algebras.
A braided 2-Lie algebra L has, for any pair of objects x, y a natural isomorphism

$$[x, y] \xrightarrow{\sim} 0$$

Every braided 2-Lie algebra is automatically symmetric.

O. Abbad, S. Mantovani, G. Metere and E.M. Vitale, Butterflies in a semi-abelian context, *arXiv* (2011).

E. Aldrovandi and B. Noohi, Butterflies I: Morphisms of 2-group stacks, *Advances in Mathematics* **221** (2009) 687–773.

G. Janelidze, Internal crossed modules, *Georgian Mathematical Journal* **10** (2003) 99–114.

A. Joyal and R. Street, Braided monoidal categories, *Macquarie Math Reports 860081* (1986)

T. Everaert, R. W. Kieboom and T. Van der Linden, Model structures for homotopy of internal categories, *T.A.C.* **15** (2005) no.3 66–94.

N. Martins-Ferreira and T. Van der Linden, A note on the "Smith is Huq" condition, *Appl. Categ. Structures* (2010).

B. Noohi, On weak maps between 2-groups, *arXiv* (2005) .

E.M. Vitale, Bipullbacks and calculus of fractions, *Cahiers de Topologie et Géométrie Différentielle Catégorique* **51** (2010) 83–113.

G.W. Whitehead, Elements of homotopy theory, *GTM* **61** (1974).

Kernels of butterflies - proof

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{H}_1(\mathbb{K}) \arrow[r] \arrow[dd] & \bullet \arrow[r, equal] \arrow[dd, "1"] & \mathcal{H}_1(\mathbb{H}) \arrow[r, equal] \arrow[dd] & \bullet \arrow[r] \arrow[d] & \mathcal{H}_1(\mathbb{G}) \arrow[dd] \\
 & & & & H \times G \arrow[dl, "\pi_H"'] \arrow[dr, "\pi_G"] \arrow[dd, "\kappa\sharp\iota"] & \\
 & H \arrow[r, "\langle 1{,}0 \rangle"] \arrow[dd, "\partial"'] & H \times G \arrow[r, "\pi_H"] \arrow[dd, "\kappa\sharp\iota"] & H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dd, "\partial"] \arrow[dl, "\iota"'] \\
 & & & & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] \arrow[dd] & \\
 & K_0 \arrow[r, "\ker\rho"'] \arrow[d] & E \arrow[r, "\sigma"'] \arrow[d] & H_0 \arrow[d] & & G_0 \arrow[d] \\
 & \mathcal{H}_0(\mathbb{K}) \arrow[r] & \bullet \arrow[r, equal] & \mathcal{H}_0(\mathbb{H}) \arrow[r, equal] & \bullet \arrow[r] & \mathcal{H}_0(\mathbb{G})
\end{tikzcd}
$$

back

Examples of butterflies

We show how to construct the weak morphism associated to a butterfly in the cases of groups, Lie algebras and Rings.

The technique

Let us consider the butterfly E in a semi-abelian algebraic variety $\mathcal{C}$, and let $U : \mathcal{C} \to \mathcal{S}$ (the axiom of choice holding in $\mathcal{S}$) a suitable "forgetful" functor. Let s be a section of σ in $\mathcal{S}$:

$$
\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
H_0 \arrow[ur, dashed, bend left, "s"] & & G_0
\end{tikzcd}
$$

The w.e. of the canonical span $\mathbb{H} \xleftarrow{\Sigma} \mathbb{E} \xrightarrow{R} \mathbb{G}$ associated to E is an equivalence in $\mathbf{Gpd}(\mathcal{S})$, so that it has a weak inverse Σ^*. The composition $\Sigma^* R$ in $\mathbf{Gpd}(\mathcal{S})$ is the weak morphism $\mathbb{H} \to \mathbb{G}$. Coherence conditions are encoded in the extension of the butterfly.

Examples of butterflies: Groups I

Let $\mathcal{C} = \mathbf{Gp}$, and $U : \mathbf{Gp} \to \mathbf{Set}_*$. Under the equivalence between crossed modules and groupoids, $\partial\colon H \to H_0$ yields the groupoid

$$G_1 = G \rtimes G_0 \underset{c}{\overset{d}{\rightrightarrows}} G_0 \qquad \text{where}$$

$$c\colon (g,x) \mapsto x, \qquad d\colon (g,x) \mapsto \partial g + x, \qquad e\colon x \mapsto (0,x).$$

Define the monoidal functor $F_E = (F_0, F_1, F_2)$:

$$\begin{aligned}
F_0 &= s\rho\colon H_0 \to G_0; \qquad x \mapsto \rho(sx) \\
F_1 &= F \rtimes F_0 \qquad \text{where} \\
& \quad\; F\colon H \to G; \qquad h \mapsto -\kappa(h) + s(\partial(h)) \\
F_2 &: \quad H_0 \times H_0 \to G_1; \qquad (x,y) \mapsto (sx + sy - s(x+y), \rho(s(x+y)))
\end{aligned}$$

Notice that, $F_2(x,y)$ is an arrow $F_0(x+y) \to F_0(x) + F_0(y)$.

Examples of butterflies: Groups II

From the classification of group extensions we know that with the short exact sequence

$$G \xrightarrow{\kappa} E \xrightarrow{\sigma} H_0$$

with a chosen set-theoretical section s of σ we can associate two functions $\alpha\colon H_0 \to \mathrm{Aut}\, G$ and $f\colon H_0 \times H_0 \to G$: with $\alpha(x)(g) = x \cdot g = sx + g - sx$ and $f(x, y) = sx + sy - s(x + y)$. Such functions satisfy the following well known relation: for any x, y, z in H_0

$$x \cdot f(y, z) + f(x, yz) = f(x, y) + f(xy, z).$$

It is now easy to show that this relation corresponds precisely to what is necessary in order to prove (associative) coherence for the monoidal functor F_E.

Examples of butterflies: Lie algebras I

A groupoid in **Lie** is called a strict Lie 2-Algebra. We consider the forgetful functor U : **Lie** $\to$ **Vect**. and we define F_E with the same technique as before.
Indeed F_0 and F_1 are defined in the same way (provided the semidirect product is performed in **Lie**!), while

$$F_2\colon (x,y) \mapsto ([sx,sy] - s[x,y], \rho(s[x,y])).$$

Examples of butterflies: Lie algebras II

From the theory of Lie algebras extensions, we know that with the extension (ι, σ) (and a linear section s of σ) is associated a linear map $\alpha \colon H_0 \to \mathbf{Der}\, G$, $\alpha(x)(g) = x \cdot g = [sx, g]$, and a bilinear skew-symmetric map $f : H_0 \times H_0 \to G$, $f(x, y) = [sx, sy] - s[x, y]$. These maps satisfy the relations

(i) for any x, y in H_0, $[\alpha(x), \alpha(y)] - \alpha([x, y]) = \mathrm{ad}_{f(x,y)}$

(ii) for any x, y, z in H_0

$$\sum_{\text{cyclic}} (x \cdot f(y, z) - f([x, y], z)) = 0$$

where ad_g is the (adjoint) action defined by $\mathrm{ad}_g(g') = [g, g']$. The first relation helps in proving the naturality of F_2, the second yields the coherence of the bracket operation with respect to the jacobian identity.

Examples of butterflies: Rings I

We call (strict) 2-*ring* a groupoid in the category of rings.
We consider the forgetful functor $U\colon \mathbf{Rng} \to \mathbf{Set}_*$. The definition of F_E goes verbatim as in the case of groups, the additive notation expressing the underlying abelian group.
The exact sequence (ι, σ) provides the data for proving that F_E is a 2-ring homomorphism.

Examples of butterflies: Rings II

In fact we use s, the set-theoretical section of σ, to define $f, \epsilon \colon H_0 \times H_0 \to G$: $f(x,y) = sx + sy - s(x+y)$, $\epsilon(x,y) = sx \cdot sy - s(x \cdot y)$, and a map $\alpha \colon H_0 \to \mathbf{Bim}G$ with $\alpha(x)(g) = (sx \cdot g, g \cdot sx)$. Then the following relations hold for any x, y, z and t in H_0

(i) $\alpha(x) + \alpha(y) - \alpha(x+y) = \mu_{f(x,y)}$
(ii) $\alpha(x) \circ \alpha(y) - \alpha(xy) = -\mu_{\epsilon(x,y)}$
(iii) $f(0,y) = 0 = f(x,0)$ and $\epsilon(0,y) = 0 = \epsilon(x,0)$
(iv) $f(x,y) + f(z,t) - f(x+z, y+t) - f(x,z) - f(y,t) + f(x+y, z+t) = 0$
(v) $-\epsilon(x,t) - \epsilon(y,t) + \epsilon(x+y,t) + f(xt,yt) - f(x,y) \cdot t = 0$
(vi) $\epsilon(t,x) + \epsilon(t,y) - \epsilon(t,x+y) - f(tx,ty) + f \cdot h(x,y) = 0$
(vii) $x \cdot \epsilon(y,z) - \epsilon(xy,z) + \epsilon(x,yz) - \epsilon(x,y) \cdot z = 0$

where μ_g is the inner bimultiplication induced by the multiplication with g

Examples of butterflies: Rings III

Now, (i) and (ii) give the naturality of F_2. Moreover, since the normalization conditions (iii) hold, the relation (iv) gives at once associative and symmetric coherence: actually for $y = 0$ we obtain the cocycle condition for the underlying (abelian) group extension, while letting $x = t = 0$ we get the symmetric coherence. Finally (vii) yields the associative coherence for the multiplication, and (v) and (vi) give the distributive coherence.

back