Alberto Facchini

Direct-sum decompositions and direct-product decompositions depending on two parameters

If M_R is a right module over a ring R, the number of ways in which M_R can be decomposed as a finite direct sum $M_1 \oplus \cdots \oplus M_n$ of submodules is extremely arbitrary. In some cases, there is uniqueness of direct-sum decomposition into indecomposables, up to a permutation (and isomorphism). This is the case, for instance, of the classical Krull-Schmidt Theorem. In other cases, the direct-sum decomposition into indecomposables is not unique up to a permutation, but depends up to two permutations. Here is an example. A module U_R is *uniserial* if the lattice of its submodules is linearly ordered under inclusion. Two modules U and V are said to have *the same monogeny class*, denoted $[U]_m = [V]_m$, if there exist a monomorphism $U \to V$ and a monomorphism $V \to U$; and they have *the same epigeny class*, denoted $[U]_e = [V]_e$, if there exist an epimorphism $U \to V$ and an epimorphism $V \to U$. Theorem: Let $U_1, \dots, U_n, V_1, \dots, V_t$ be $n+t$ non-zero uniserial right modules over a ring R. Then the direct sums $U_1 \oplus \cdots \oplus U_n$ and $V_1 \oplus \cdots \oplus V_t$ are isomorphic R-modules if and only if $n = t$ and there exist two permutations σ and τ of $\{1, 2, \dots, n\}$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i = 1, 2, \dots, n$. Other classes of modules follow the same pattern.

There's no reason why only modules should have this behavior, why only some classes of modules should follow this pattern. Thus we have studied direct-product decompositions of the analogous classes of groups, but, in the cases we have studied, we have always find uniqueness of direct-product decomposition into indecomposables. In the study of direct-product decompositions of a group G, we have seen that the proper categorical setting to work in is that of the category G-**Grp** of G-groups, whose objects are the groups H on which G acts via a group morphism $G \to \mathrm{Aut}(H)$.