Categorical aspects of monoid extensions and actions

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(joint work with D. Bourn, N. Martins-Ferreira and A. Montoli)

The notions of protomodular [1] and semi-abelian [6] categories provide the appropriate categorical setting to describe the essential features of groups and many other algebraic structures. The main goal of these talks is to present a convenient categorical context for weaker algebraic structures like the one of monoids.

The well known equivalence between group actions and split extensions does not hold for monoids. There we have, on one hand, (classical monoid) actions/semidirect products and, on the other hand, internal actions/ categorical semidirect products, as defined in [2], and they are not equivalent constructions, in general. This is also the case of the well known equivalence between crossed modules and internal categories in the category of groups, that it is not true for monoids.

Monoid actions correspond, via the semidirect product of monoids, to a special class of split epimorphisms in this category, a class identified in [7] where they are called *Schreier split epimorphisms*. The Schreier split epimorphisms allow to recover, in the category of monoids, partial results of protomodular as well as of Mal'tsev categories ([4]), and this can be extended to categories of *monoids with operations* which include semilattices, semirings and several other algebraic structures.

In particular, in monoids, the *special Schreier extensions* (a notion introduced in [3]) with fixed abelian kernel can be equipped with an abelian group structure like in the case of all group extensions with abelian kernel in the category of groups.

We describe a natural generalization: the pointed *S-protomodular categories*, where S is a suitable class of split epimorphisms, a notion introduced in [3] and studied in [5]. These categories satisfy, relatively to the class S, many properties of Mal'tsev and protomodular categories, namely every S-split epimorphism gives rise to a short exact sequence, the split short five lemma for the S-split exact sequences holds or the fact that a reflexive S-relation is transitive. S-categories, S-groupoids and S-special morphisms are amongst the notions that we intend to present as well as some other relative results.

References

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