CatAlg2018

Workshop in Categorical Algebra

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Abstracts of the talks

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Extraction of a Σ-Mal'tsev (resp. Σ-protomodular) structure from a Mal'tsev (resp. protomodular) subcategory

Suppose Σ is a point-congruous class of split epimorphisms in a category $\mathbb{D}$. If $\mathbb{D}$ is Σ-Mal'tsev (resp. Σ-protomodular) category, it produces a Mal'tsev (resp. protomodular) subcategory $\sharp\mathbb{D} \hookrightarrow \mathbb{D}$ [1].

The aim of this work is kind of converse: given a fully faithfull inclusion $j : \mathbb{C} \hookrightarrow \mathbb{D}$, where $\mathbb{C}$ is a Mal'tsev (resp. protomodular) subcategory, to find conditions on this inclusion such that it produces Σ-Mal'tsev (resp. Σ-protomodular) structure on $\mathbb{D}$.

References:

[1] D. Bourn, *Partial linearity and partial natural Mal'tsevness, Theory and Applications of Categories, 31, 2016, 418-443.*

Alan S. Cigoli*

Università degli Studi di Milano - Université catholique de Louvain

Fibered aspects of Yoneda's regular span

In his pioneering 1960 paper [3], Nobuo Yoneda presents a formal categorical setting in order to formulate the classical theory of $\mathbf{Ext}^n$ functors avoiding the request of having enough projectives. He studied, in an additive category, the functor sending any exact sequence of length n

$$0 \longrightarrow b \xrightarrow{j} e_n \longrightarrow e_{n-1} \longrightarrow \cdots \qquad \longrightarrow e_1 \xrightarrow{p} a \longrightarrow 0$$

to the pair (a, b).

The basic observation is that, thanks to the dual properties of pushouts and pullbacks, it is possible to define translations and cotranslations of exact sequences. This analysis leads Yoneda to identify a set of formal properties of a functor

$$S\colon \mathcal{X} \to \mathcal{A} \times \mathcal{B}$$

in order to get the axioms of what he calls a *regular span.*

In fact, his Classification Theorem in [3, §3.2] follows in a purely formal way from the axioms of regular span, once one considers connected components of the fibers of S (called *similarity classes* in [*loc. cit.*]). We can interpret his result by saying that, with any regular span S, it is possible to associate what is nowadays called (see [2]) a *two-sided discrete fibration* $\bar{S}\colon \bar{\mathcal{X}} \to \mathcal{A} \times \mathcal{B}$, together with a factorization of S through a functor $Q\colon \mathcal{X} \to \bar{\mathcal{X}}$.

In this talk, I will show that Yoneda's notion of a regular span $S\colon \mathcal{X} \to \mathcal{A} \times \mathcal{B}$ can be interpreted as a special kind of morphism, that we call *fiberwise opfibration*, in the 2-category $\mathbf{Fib}(\mathcal{A})$. We study the relationship between these notions and those of internal opfibration and two-sided fibration. This fibrational point of view makes it possible to interpret Yoneda's Classification Theorem as the result of a canonical factorization, and to extend it to a non-symmetric situation, where the fibration given by the product projection $Pr_0\colon \mathcal{A} \times \mathcal{B} \to \mathcal{A}$ is replaced by any split fibration over $\mathcal{A}$. This new setting allows us to transfer Yoneda's theory of extensions to the non-additive analog given by crossed extensions for the cases of groups (see [1]) and other algebraic structures.

References:

[1] J. Huebschmann, Crossed n-fold extensions of groups and cohomology, Comment. Math. Helv. 55 (1980) 302–313.

[2] R. Street, Fibrations and Yoneda's lemma in a 2-category, in: A. Dold, B. Eckmann (Eds.), Category Seminar, Lecture Notes in Mathematics, vol. 420, Springer, Berlin, 1974, pp. 104–133

[3] N. Yoneda, On Ext and exact sequences. J. Fac. Sci. Univ. Tokyo Sect. I 8 (1960) 507–576.

Maria Manuel Clementino[†]

Universidade de Coimbra

On the categorical behaviour of ordered groups

In this talk we explore both the topological and algebraic features of the category of (pre)ordered groups. Our results stand on the categorical properties of both the category of topological groups, and of groups and monoids.

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[*]Joint work with S. Mantovani, G. Metere and E. M. Vitale.

[†]Joint work with Andrea Montoli and Nelson Martins-Ferreira

Alberto Facchini
Università degli Studi di Padova

An introduction to cotorsion pairs, covering classes and tilting modules

We plan to give an easy short introduction to the notions of cotorsion pairs, covering classes and tilting modules. In particular, we will focus on the relations between these notions, which have played an important role in the development of module theory and representation theory in the last twenty years.

Diana Rodelo[‡]
Universidade do Algarve

Observations on the Shifting Lemma

We prove that Mal'tsev and Goursat categories may be characterised through (stronger) variations of the Shifting Lemma. More precisely, a regular category $\mathbb{C}$ is a Mal'tsev category if and only if the Shifting Lemma holds for reflexive relations R, S, and T in $\mathbb{C}$. And a regular category $\mathbb{C}$ is a Goursat category if and only if the Shifting Lemma holds for a reflexive relation S and reflexive and positive relations R and T in $\mathbb{C}$.

Xabier García-Martínez
Universidade de Santiago de Compostela

A characterisation of Lie algebras via algebraic exponentiation

In this talk we will describe the variety of Lie algebras via algebraic exponentiation, a concept introduced by James Gray in his Ph.D. thesis. We will prove that the variety of Lie algebras over a field K of characteristic zero is the only non-abelian variety of non-associative algebras over K which is locally algebraically cartesian closed (LACC). We will also extend this result to varieties of n-algebras, and we will discuss what happens in prime characteristic.

[‡]Joint work with Marino Gran and Idriss Tchoffo Nguefeu

Marino Gran[§]
Université catholique de Louvain

New interactions between categorical algebra and Hopf algebra theory

Some properties of the category of cocommutative Hopf algebras have been recently explored from the perspective of categorical algebra [2, 4, 6]. Most of the results in this direction have been obtained under the assumption that the characteristic of the base field K is zero. In the present work with F. Sterck and J. Vercruysse we drop this assumption, and establish some properties of the category $\mathsf{Hopf}_{K,coc}$ of cocommutative Hopf algebras over an arbitrary field K.

In this talk we prove that the category $\mathsf{Hopf}_{K,coc}$ is semi-abelian and action representable. This result can be seen as a non-abelian version of Takeuchi's classical theorem asserting that the category of commutative and cocommutative Hopf algebras is abelian [5]. We shall then make a few remarks on the notion of *center* in $\mathsf{Hopf}_{K,coc}$ [1, 3], and give an example of a localization in the semi-abelian category $\mathsf{Hopf}_{K,coc}$. Some possible developments in categorical Galois theory will also be discussed.

References:

[1] N. Andruskiewitsch, *Notes on extensions of Hopf algebras*, Canad. J. Math. 48 (1), 1996, 3-42.

[2] M. Gran, G. Kadjo, and J. Vercruysse, *A torsion theory in the category of cocommutative Hopf algebras*, Appl. Categ. Struct. 24 (3), 2016, 269-282.

[3] M. Gran, G. Kadjo, and J. Vercruysse, *Split extension classifiers in the category of cocommutative Hopf algebras,* Bull. Belgian Math. Society Simon Stevin, 2018, accepted for publication.

[4] X. García Martinez and T. Van der Linden *A note on split extensions of bialgebras*, Forum Mathematicum, 2018, published online.

[5] M. Takeuchi, A correspondence between Hopf ideals and sub-Hopf algebras, *Manuscripta Math.* 7, 1972, 252-270.

[6] C. Vespa and M. Warmst, *On some properties of the category of cocommutative Hopf algebras*, North-Western European J. of Math. 4, 2018, 21-37.

Pierre-Alain Jacqmin
University of Ottawa

Internal fibrations and their normalizations

This is a join work with Sandra Mantovani, Giuseppe Metere and Enrico Vitale. We characterize fibrations between internal groupoids in terms of the comparison

[§]Joint work with Florence Sterck and Joost Vercruysse functor from a pullback to the corresponding strong homotopy pullback. A similar characterization also holds for $*$-fibrations using kernels and strong homotopy kernels. As an application, we show how to derive the Brown exact sequence from the Gabriel-Zisman sequence. We then focus on the corresponding notions of fibrations and $*$-fibrations in the category of arrows and how they interact with the normalization functor. In a similar way than for groupoids, this allows us to explain how one can derive, from the snail lemma, (a generalized form of) the snake lemma.

George Janelidze
University of Cape Town

What is Galois theory of a factorization system?

The talk is devoted to the Galois theory of the reflection $\mathbb{C} \to \mathbb{X}$, in which: $\mathbb{X}$ is a category with pullbacks equipped with a factorization system (E, M); $\mathbb{C}$ is the full subcategory of the arrow category of $\mathbb{X}$ with objects all elements of E; $\mathbb{X}$ is considered as a full subcategory of $\mathbb{C}$ by identifying its objects with their identity morphisms.

Zurab Janelidze
University of Stellenbosch

Some questions on the logic of categorical algebra

In this talk I will give an overview of some of the topics in categorical algebra that are interesting from a logical perspective, arriving in each case to some open questions. These questions concern mostly the study of exactness properties, their relation to Mal'tsev conditions in universal algebra, and analysis of the techniques for deriving consequences from the exactness properties.

Andrea Montoli¶
Università degli Studi di Milano

Homological lemmas in Σ-Mal'tsev and in Σ-protomodular categories

The classical homological lemmas are a widely studied topic in categorical algebra. It is known that, for pointed, finitely complete categories, the validity of the split short five lemma is equivalent to Bourn-protomodularity. In the context of normal categories, Z. Janelidze proved in [5] that the upper and the lower nine lemma, that are equivalent, hold if and only if the category is subtractive. The validity of the denormalized version of the nine lemma, replacing exact sequences with exact forks, is equivalent to the fact that the ground category is Goursat, as proved by M. Gran and D. Rodelo in [4].

The category of monoids is not subtractive, but restricted versions of the above mentioned homological lemmas hold in it, if we consider *special Schreier extensions* [2, 6]. In order to understand, from a categorical point of view, the homological properties of monoids, the notions of Σ-protomodular [3] and Σ-Mal'tsev [1] categories, with respect to a pullback stable class Σ of points, have been introduced. These categories have, relatively to the class Σ, many of the typical properties of protomodular and Mal'tsev categories.

In this talk we will discuss the validity of the classical homological lemmas in Σ-protomodular and Σ-Mal'tsev categories, describing the stability conditions on the class Σ that are necessary for the homological lemmas to hold. In particular, a strong asymmetry between the upper and the lower nine lemma will appear, in contrast with the "absolute" case (i.e. the case when the class Σ is the class of all points), where they are always equivalent.

References:

[1] D. Bourn, *Partial Mal'tsevness and partial protomodularity*, preprint arXiv:1507.02886v1, 2015.

[2] D. Bourn, N. Martins-Ferreira, A. Montoli, M. Sobral, *Schreier split epimorphisms in monoids and in semirings*, Textos de Matemática (Série B), Departamento de Matemática da Universidade de Coimbra, vol. 45 (2013).

[3] D. Bourn, N. Martins-Ferreira, A. Montoli, M. Sobral, *Monoids and pointed S-protomodular categories*, Homology, Homotopy Appl. 18 n.1 (2016), 151-172.

[4] M. Gran, D. Rodelo, *A new characterization of Goursat categories*, Apll. Categ. Structures 20 (2012), 229-238.

[5] Z. Janelidze, *The pointed subobject functor,* 3×3 *lemmas and subtractivity of spans*, Theory Appl. Categ. 23 (2010), 221-242.

[6] N. Martins-Ferreira, A. Montoli, M. Sobral, *The Nine Lemma and the push for-*

¶Joint work with Dominique Bourn

ward construction for special Schreier extensions of monoids, submitted, preprint DMUC 16-17, 2016.

Walter Tholen
York University, Toronto

Factorizations Then and Now

Factorization systems have been of interest in category theory since its very beginnings, first as a means to axiomatize special classes of morphisms, such as subobject injections and quotient object projections, or fibrations, cofibrations and weak equivalences in homotopy theory, and then as an additional structure of a category, facilitating the proofs of certain theorems, normally by augmenting the category?s completeness or cocompleteness properties. Recent works on enriched and higher-dimensional systems and on so-called algebraic weak factorization systems show that the subject continues to grow, deepen and extend its reach.

However, in contrast to the standard categorical notions, such as that of limit or adjunction, there seems to be no generally adopted notion of factorization system which would fit all purposes. Special conditions, such as a restriction to epi-mono-factorizations, assumed for convenience in one area, may prove to be rather forbidding in another and, in any case, will likely obscure the true nature of the notion.

In this talk I will give an overview of the principal types of categorical factorization notions and exhibit their relationships with the standard notions of category theory, and with each other. I will pay special attention to

- forgotten or ignored contributions which nevertheless may deserve priority credit over frequently cited contributions and serve as a resource for previously overlooked facts;
- being careful about any unneeded hypotheses which may appear convenient in one context but harmful in another;
- treating orthogonal and weak systems in concert and reconciling the "class-of-morphism-approaches" with functorial approaches;
- various enriched and higher-dimensional notions of factorization system, as well as generalizations of single-morphism factorizations;
- emphasizing the fundamental role of fibrations for factorization systems.

Time permitting, I will discuss in particular factorization systems in partial-morphism categories.

Benno van den Berg[‖]
Universiteit van Amsterdam

Path categories

The purpose of this talk is to introduce the notion of a path category (short for a category with path objects). Like other notions from homotopical algebra, such as a category of fibrant objects or a Quillen model structure, it provides a setting in which one can develop some homotopy theory. For a logician this type of category is interesting because it provides a setting in which many of the key concepts of homotopy type theory (HoTT) make sense. Indeed, path categories provide a syntax-free way of entering the world of HoTT, and familiarity with (the syntax of) type theory will not be assumed in this talk. Instead, I will concentrate on basic examples and results.

[‖]Partly based on joint work with Ieke Moerdijk