NEW TRENDS IN HOPF ALGEBRAS AND MONOIDAL CATEGORY

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Groupal Pseudofunctors

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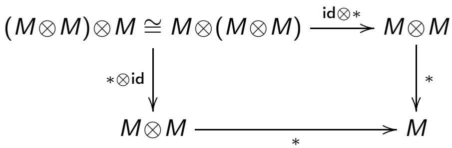
Introduction
Internal algebraic structures in a category

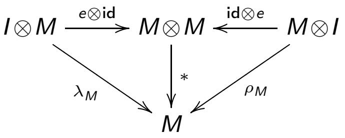
Internal monoids

One good reason to study monoidal categories:

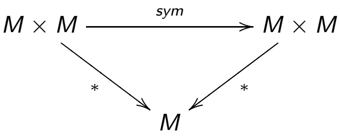
you can define **internal monoid** *objects*

$M = (M,\ M \otimes M \xrightarrow{*} M\ ,\ I \xrightarrow{e} M\ )$ monoid object in $\mathbf{B} = (\mathbf{B}, \otimes, I, \alpha, \lambda, \rho)$:





M is **commutative** if endowed with *sym*: $M \times M \to M \times M$ s.t.



Examples:

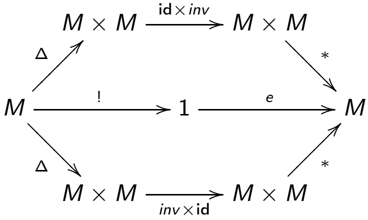
- $\mathbf{B} = (\mathbf{Set}, \times)$, M is a monoid
- $\mathbf{B} = (\mathbf{Gp}, \times)$, or $\mathbf{B} = (\mathbf{Ab}, \times)$ M is an abelian group
- $\mathbf{B} = (\mathbf{Ab}, \otimes)$ M is a ring (with unit)

Internal groups

If **B** *is* **cartesian monoidal** $(\otimes = \times, I = 1)$*:*

you can define **internal (abelian) group** *objects*

The monoid $(M, *, e)$ is a group if endowed with $inv \colon M \to M$ s.t.



Lax monoidal structures

Fact: lax monoidal functors take monoids to monoids

Special case: **B** with fin. prod. and $F\colon \mathbf{B} \to \mathbf{Set}$ preserving them,

- F takes internal monoids in **B** to monoids (in **Set**):

$$\begin{array}{c} M \times M \\ \downarrow{\scriptstyle *} \\ M \end{array} \quad \mapsto \quad \begin{array}{c} F(M) \times F(M) \\ \downarrow{\scriptstyle F(*)} \\ F(M) \end{array}$$

$$\begin{array}{ccc} \mathbf{Ab}(\mathbf{B}) & \xrightarrow{\mathbf{Ab}(F)} & \mathbf{Ab} \\ \downarrow & & \downarrow \\ \mathbf{Gp}(\mathbf{B}) & \xrightarrow{\mathbf{Gp}(F)} & \mathbf{Gp} \\ \downarrow & & \downarrow \\ \mathbf{Mon}(\mathbf{B}) & \xrightarrow{\mathbf{Mon}(F)} & \mathbf{Mon} \\ \downarrow & & \downarrow \\ \mathbf{B} & \xrightarrow{F} & \mathbf{Set} \end{array}$$

i.e. F lifts to **Mon**(**B**).

With same hyps, F lifts to **Gp**(**B**) and to **Ab**(**B**).

Consequence: if $\mathbf{Ab}(\mathbf{B}) \to \mathbf{B} = \mathrm{id}$ (e.g. when **B** is abelian), then F factors through the forgetful functor $U\colon \mathbf{Ab} \to \mathbf{Set}$.

Today's plan: push these results one dimension up!

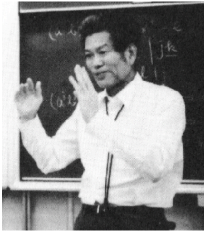
Motivating example: Baer sums *à la* Yoneda

Let **B** be an abelian category, and A in **B**. A functor is defined [Yoneda, 1960]:

$$Ext^n(A,-)\colon \mathbf{B} \to \mathbf{Set}$$

$$B \mapsto \{[0 \to B \to E_n \to \cdots \to E_1 \to A \to 0]_\sim\}$$

where $\sim$ is the **connectedness** relation, i.e. the equiv. rel. generated by maps of n-extensions that fix B and A.



Now, $Ext^n(A,-)$ preserves finite products, and $\mathbf{Ab}(\mathbf{B}) = \mathbf{B}$,

$\Rightarrow$ we get the factorization $Ext^n(A,-)\colon \mathbf{B} \to \mathbf{Ab}$.

Fact: The abelian group structure induced on $Ext^n(A,B)$ is that of Baer sums.

WHAT IF we **do not** *take the quotient on the connectedness relation?*

Get a pseudofunctor $EXT^n(A,-)\colon \mathbf{B} \to \underline{\mathbf{Cat}}$. [dictionary: *pseudo* = *up-to-iso*]

A natural question is: under which conditions does such pseudofunctor induce any kind of structure on the categories $EXT^n(A,B)$. Monoidal? Groupal?

Part 1

Internal weak structures in a 2-category

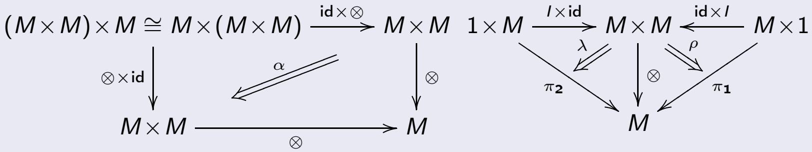
Internal pseudomonoid objects

One good reason to study monoidal 2-categories:

you can define **internal pseudomonoid** *objects*

Definition

A pseudomonoid in 2-category $\underline{\mathbf{B}}$ with finite products is an object M endowed with 1-cells $\otimes\colon M \times M \to M$, $I\colon 1 \to M$ and (coherent) iso 2-cells:



Examples:

- A **monoidal category** $\mathbf{C} = (\mathbf{C}, \otimes, I, \alpha, \lambda, \rho)$ is a pseudomonoid in $\underline{\mathbf{Cat}}$.
- Let $\mathbf{B}$ be a category with fin. prod. considered as a 2-category with trivial 2-cells. A pseudomonoid M in $\mathbf{B}$ is just an ordinary internal monoid.

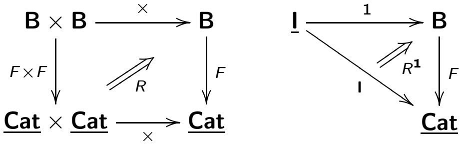
Lax monoidal pseudofunctors

Let **B** be a category with fin. prod. considered as a loc. disc. 2-category.

A **pseudofunctor** $F\colon \mathbf{B} \to \underline{\mathbf{Cat}}$ is a weak 2-functor which preserves composition and identities only up to coherent isomorphisms:

$$\phi_{g,f}\colon F(g)\circ F(f) \cong F(g\circ f) \qquad \phi^{\mathbf{1}}\colon \mathrm{id}_{F(B)} \to F(\mathrm{id}_B)$$

$F\colon (\mathbf{B},\times,1) \to (\underline{\mathbf{Cat}},\times,\mathbf{I})$ is **lax (2-)monoidal** if it is endowed with with pseudonatural transformations:

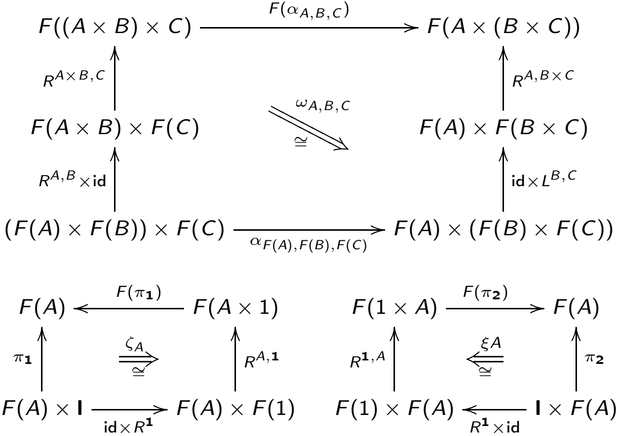


with functor components:

$$R^{A,B}\colon F(A)\times F(B) \to F(A\times B) \qquad R^{\mathbf{1}}\colon \mathbf{I} \to F(1)$$

and suitable modifications with components that do not fit this page...

...but do fit **this** page...



...together with some coherence conditions that do not fit this page and will not be commented here!

Cartesian monoidal pseudofunctors

Good news: Monoidal pseudofunctors take pseudomonoids to pseudomonoids.

[Day Street, 1997]

In particular for monoidal

$$F\colon (\mathbf{B}, \times, 1) \to (\underline{\mathbf{Cat}}, \times, \mathbf{I})$$

- F takes (commutative) monoids in $\mathbf{B}$ to (symmetric) monoidal categories,

$$\boxed{\begin{array}{c} M \times M \\ \Big\downarrow * \\ M \end{array}} \quad \mapsto \quad \boxed{\begin{array}{c} F(M) \times F(M) \\ \Big\downarrow R^{M,M} \\ F(M \times M) \\ \Big\downarrow F(*) \\ F(M) \end{array}}$$

i.e. F lifts to $\mathbf{psMon}(\mathbf{B}) = \mathbf{Mon}(\mathbf{B})$.

$$\begin{array}{ccc} \mathbf{Ab}(\mathbf{B}) & \dashrightarrow & \boxed{???} \\ \downarrow & & \downarrow \\ \mathbf{Gp}(\mathbf{B}) & \dashrightarrow & \boxed{???} \\ \downarrow & & \downarrow \\ \mathbf{Mon}(\mathbf{B}) & \xrightarrow{\mathbf{psMon}(F)} & \underline{\mathbf{MonCat}} \\ \downarrow & & \downarrow \\ \mathbf{B} & \xrightarrow{F} & \underline{\mathbf{Cat}} \end{array}$$

Question: does F lift to groups and to abelian groups?

The notion we need to fill the $\boxed{???}$'s is that of **internal pseudogroups** in $\underline{\mathbf{Cat}}$.

Internal pseudogroups

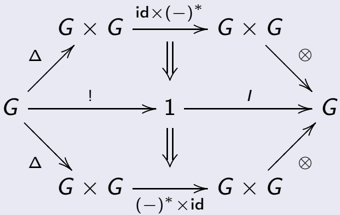
Internal pseudogroups have been introduced in [Baez Lauda, 2004], with the name of **weak 2-groups**.

Definition

A pseudogroup in a 2-category with finite products is a pseudomonoid

$$(G, \otimes\colon G \times G \to G, I\colon 1 \to G, \dots)$$

endowed with an *inverse* 1-cell $(-)^*\colon G \to G$ and iso 2-cells



$+$ coherence

In fact pseudogroups existed already *in nature* well before 2004.

Categorical groups *aka* weak 2-groups *aka* *Gr*-catégories

Pseudogroups in **Cat** have been studied by Grothendieck's student **Hoáng Xuân Sính** in her 1975 PhD thesis, who named them *Gr*-catégories.
They are **monoidal groupoids** with every objects pseudoinvertible w.r.t. tensor product.

Why monoidal groupoids, and not just monoidal categories?

Fact: If **G** is a pseudogroup in **Cat**, then its 1-cells are invertible w.r.t. composition, i.e. **G** is a groupoid. ??? [Baez Lauda, 2004]

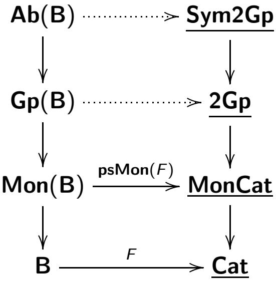
This happens because we want $(-)^* : \mathbf{G} \to \mathbf{G}$ *to be internal, i.e. a* covariant *functor. Requiring just pseudo invertible objects in a monoidal category produces a contravariant functor.*

Example: Let **PreOrd** be the cat. of preorders, seen as a 2-cat. A preordered group G is an internal pseudomonoid in **PreOrd** which happens to be a group. Inversion map is antitone, i.e. contravariant. If we impose inversion map of G to be monotone, then the underlying preorder is an equivalence relation.

So far so good...

We found very good candidates to fill the ??? 's.

Now we have to fill the ⇢ 's!



In dimension 1, every product preserving functor takes internal (abelian) groups to (abelian) groups.

Do *all* cartesian monoidal pseudo-functors take (abelian) groups to (Symmetric) 2-groups?

The answer is: **NO!**

Example: Consider the lax monoidal pseudofunctor

$$\mathrm{Sub}(-)\colon (\mathbf{Ab}, \oplus, 0) \to (\underline{\mathbf{Cat}}, \times, \mathbf{I})$$

that assigns to every abelian group A the poset $\mathrm{Sub}(A)$ of its subobjects. The canonical abelian group structure on an object A induces the symmetric monoidal structure on $\mathrm{Sub}(A)$ given by the join of subobjects. However, $\mathrm{Sub}(A)$ is not a groupoid, hence it cannot support any 2-group structure.

Part 2

Preservation of group-like structure

Lax monoidal pseudofunctors

The example of Sub$(-)$ makes the following definition sensible.

Definition

A lax monoidal pseudofunctor $F\colon (\mathbf{B}, \times, I) \to (\underline{\mathbf{Cat}}, \times, \mathbf{I})$ is termed **groupal** if it lifts to a pseudofunctor $\hat{F}$ that makes the diagram commute:

$$\begin{array}{ccc} \mathbf{Gp}(\mathbf{B}) & \longrightarrow & \underline{\mathbf{2Gp}} \\ \downarrow & & \downarrow \\ \mathbf{B} & \xrightarrow[F]{} & \underline{\mathbf{Cat}} \end{array}$$

Fact: $\hat{F}$ clearly restricts to $\mathbf{Ab}(\mathbf{B}) \to \underline{\mathbf{Sym2Gp}}$.

Aim of the second part of my talk: Characterize and (perhaps) explain groupal pseudofunctors.

Pseudofunctor vs (op)Fibrations.

There is a canonical way to translate a pseudofunctors into Grothendieck fibrations: the Grothendieck-Bénabou construction.

Bénabou's viewpoint is well-known:

> *[...] one might feel forced to accept pseudo–functors and the ensuing bureaucratic handling of "canonical isomorphisms". However, as we will show immediately one may replace pseudo–functors* $H\colon \mathbf{B}^{op} \to \underline{\mathbf{Cat}}$ *by fibrations* $P\colon \mathbf{X} \to \mathbf{B}$ *where this bureaucracy will turn out as luckily hidden from us.* [Streicher, 2022]

This is also [Cigoli, Mantovani, M., 2022]'s take on the subject, where results are achieved by fibrational techniques.

On the other hand, by using the language pseudofunctors, **it is easier to focus on the dimension leap**, which is a main theme in my talk. Therefore, I will keep walking on the pseudofunctor side, and try to hide fibrations under the carpet!

However, fibrational details can be found in the cited article ;)

Proposition (Cigoli, Mantovani, M., 2022)

Let $\mathbf{B}$ *be a category with fin. prod., and* $F\colon \mathbf{B} \to \underline{\mathbf{Cat}}$ *be a pseudofunctor. Then* F *is canonically endowed with an* **oplax** *symmetric monoidal structure*

$$\begin{array}{ccc} \mathbf{B}\times\mathbf{B} & \xrightarrow{\times} & \mathcal{B} \\ {\scriptstyle F\times F}\downarrow & \swarrow_{L} & \downarrow{\scriptstyle F} \\ \underline{\mathbf{Cat}}\times\underline{\mathbf{Cat}} & \xrightarrow[\times]{} & \underline{\mathbf{Cat}} \end{array} \qquad \begin{array}{ccc} \underline{\mathbf{I}} & \xrightarrow{I} & \mathbf{B} \\ & \searrow^{\mathbf{I}} \;\; \swarrow_{L^1} & \downarrow{\scriptstyle F} \\ & & \underline{\mathbf{Cat}} \end{array}$$

$$L^{A,B}\colon F(A\times B) \to F(A)\times F(B)$$
$$Z \mapsto (F(\pi_1)(Z), F(\pi_2)(Z))$$

$$L^1\colon F(I) \to \mathbf{I}$$
$$J \mapsto *$$

Theorem (Cigoli, Mantovani, M., 2022)

Let $\mathbf{B}$ *be a cat. with fin. prod., and* $F\colon \mathbf{B} \to \underline{\mathbf{Cat}}$ *be a pseudofunctor. TFAE:*

- *Pseudonat. transf.* L^1 *and* L *have a right adjoints* R^1 *and* R *in the hom-2-cats* $\mathbf{PsFunct}(\underline{\mathbf{I}}, \underline{\mathbf{Cat}})$ *and* $\mathbf{PsFunct}(\mathbf{B}\times\mathbf{B}, \underline{\mathbf{Cat}})$ *respectively.*
- F *is* **cartesian**, *i.e. endowed with a lax symmetric monoidal structure*

$$(F, R, R^1, \cdots)\colon (\mathbf{B}, \times, I) \longrightarrow (\underline{\mathbf{Cat}}, \times, \mathbf{I})$$

s.t. $R^{A,B}(X, Y) = X \times Y$ *and* $R^1(\star) = 1$.

Proposition (Cigoli, Mantovani, M., 2022)

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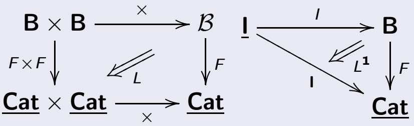

$$L^{A,B}\colon F(A\times B) \to F(A)\times F(B)$$
$$Z \mapsto (F(\pi_1)(Z), F(\pi_2)(Z))$$

$$L^1\colon F(I) \to \mathbf{I}$$
$$J \mapsto *$$

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- F *is* **cartesian**, *i.e. endowed with a lax symmetric monoidal structure*

$$(F, R, R^1, \cdots)\colon (\mathbf{B}, \times, I) \longrightarrow (\underline{\mathbf{Cat}}, \times, \mathbf{I})$$

s.t. $R^{A,B}(X, Y) = X \times Y$ *and* $R^1(\star) = 1$. *in* $\mathbf{X} = \int_{\mathbf{B}} F$

Theorem (Cigoli, Mantovani, M., 2022)

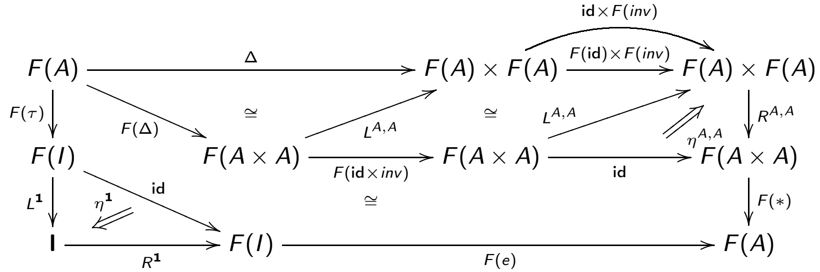
Let **B** *be a category with fin. prod., and* $F\colon \mathbf{B} \to \underline{\mathbf{Cat}}$ *be a cartesian (lax symmetric monoidal) pseudofunctor. TFAE:*

- *For every* A *in* **B** *supporting an internal group structure, the units*

$$\eta^{\mathbf{1}}\colon id_{F(I)} \Rightarrow R^{\mathbf{1}} \circ L^{\mathbf{1}} \qquad \eta^{A,A}\colon id_{F(A\times A)} \Rightarrow R^{A,A} \circ L^{A,A}$$

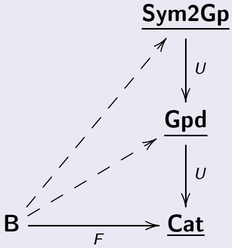
  of the adjunctions are isomorphisms.
- *The pseudofunctor* F *is groupal.*

Idea of the proof.



Corollary (Cigoli, Mantovani, M., 2022)

Let $\mathbf{B}$ *be a category with fin. prod. such that* $\mathbf{Ab}(\mathbf{B}) = \mathbf{B}$*, and* $F\colon \mathbf{B} \to \underline{\mathbf{Cat}}$ *be a cartesian (lax symmetric monoidal) pseudofunctor. The following factorizations imply one another:*



Back to Yoneda

Fix an object A of an abelian category $\mathbf{B}$ and recall the pseudofunctor

$$EXT^n(A,-)\colon \mathbf{B} \to \underline{\mathbf{Cat}}$$

of n-fold extensions of A.

- $n = 1$. For every B in $\mathbf{B}$, each $EXT^1(A, B)$ is a groupoid.
  $\Rightarrow$ *The pseudofunctor is groupal, and B induces a symmetric 2-group structure on $EXT^1(A, B)$. Its π_0 is the cohomology group $H^2(A, B)$.*
- $n > 1$. The categories $EXT^n(A, B)$ are not groupoids, in general.
  $\Rightarrow$ *The pseudofunctor is not groupal and B only induces a symmetric monoidal structure on $EXT^n(A, B)$.*

However, even if they are not, they **can be made groupoids** *by taking suitable* **categories of fractions***, and recover the cohomology groups $H^n(A, B)$, $n > 2$.*

In the same way as for the abelian case of $EXT^n(A,-)$, one can deal with **non abelian cohomology** *by means of* **crossed n-fold extensions** *in a strongly semi-abelian category.*

But then, it becomes hard not to deal with the fibrational POV...

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THANK YOU FOR YOUR ATTENTION!