Extension theory and the calculus of butterflies

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Overview

- Introduction
- Internal crossed modules and butterflies
- Extension theory

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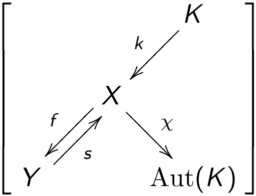
Intro: the tale of group extensions

Let K and Y be groups. It is a classical result that the set of (eq cls of) split extensions is in bijections with the set of Y-actions on K.

$$\mathrm{SpExt}(Y, K) \qquad \cong \qquad \mathbf{Gp}(Y, \mathrm{Aut}(K))$$



$$\mapsto \qquad Y \xrightarrow{\chi \cdot s} \mathrm{Aut}(K)$$

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When the extension $K \xrightarrow{k} X \xrightarrow{f} Y$ is no longer split, the homomorphism s fails to exist.

Still, one can find a set-theoretical section s'

but $\chi \cdot s'$ is no longer an action.

It is a *weak action*, in the sense of [Blanco, Bullejos and Faro, 2005].

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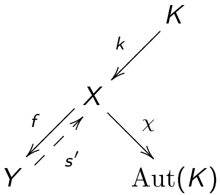

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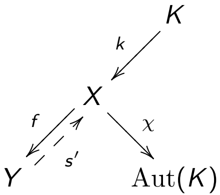

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This way, we extend the equivalence

$$\mathrm{SPEXT}(Y, K) \simeq \mathbf{Gp}(Y, \mathrm{Aut}(K))$$

to the equivalence

$$\mathrm{EXT}(Y, K) \simeq \mathbf{CG}(D(Y), \mathrm{AUT}(K))$$

The group $\mathrm{Aut}(K)$ underlies the (strict) monoidal groupoid

$$\mathrm{AUT}(K) \quad = \quad \begin{array}{c} K \rtimes \mathrm{Aut}(K) \\ d \downarrow \uparrow \downarrow c \\ \mathrm{Aut}(K) \end{array}$$

and the map $\chi \cdot s'$ underlies a monoidal functor $D(Y) \longrightarrow \mathrm{AUT}(K)$, where $D(Y)$ is the discrete monoidal groupoid associated with Y.

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Intro: the other tale of group extensions

Let's turn our attention to the internal structure of of the set $\mathrm{Ext}(Y, K)$.
Its description is obtained from Schreier-Mac Lane extension theorem.

With any extension (k, f) is associated an *abstract kernel* ϕ

$$\begin{array}{ccccc} K & \xrightarrow{\;k\;} & X & \xrightarrow{\;f\;} & Y \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \exists ! \phi} \\ \mathrm{Inn}(K) & \longrightarrow & \mathrm{Aut}(K) & \longrightarrow & \mathrm{Out}(K) \end{array}$$

that induces a Y-action $\bar{\phi}$ on $Z(K)$, the centre of the group K.

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Theorem (Schreier - Mac Lane)

Let $\mathrm{OpExt}(Y, K, \phi) \neq \emptyset$ *be the set of extensions of the same* ϕ.

- $H^2_\phi(Y, Z(K))$ *operates simply and transitively on* $\mathrm{OpExt}(Y, K, \phi)$

$\Rightarrow$ *there is a bijection* $\mathrm{OpExt}(Y, K, \phi) \cong H^2_\phi(Y, Z(K))$

Given a morphism $Y \xrightarrow{\phi} \mathrm{Out}(K)$,

- $\mathrm{OpExt}(Y, K, \phi) \neq \emptyset$ *if, and only if,* $[\phi] = 0$

where $[\phi] \in H^3_\phi(Y, Z(K))$ *is uniquely determined by* ϕ.

$$\mathrm{Ext}(Y, K) = \coprod_\phi \mathrm{OpExt}(Y, K, \phi)$$

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An intrinsic version of Schreier-Mac Lane theory has been introduced by Bourn in 2008 for exact pointed protomodular action representative categories.

It has been further developed by Bourn, Montoli (2012), and by Cigoli, M., Montoli (2013), for action accessible categories.

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First: *extensions are (classified by some) weak monoidal functors*

Second: *Extensions are classified by H^2_ϕ-actions*

The aim of my talk is show how the theory of classification of group extensions is an instance of a more general one where these two viewpoints are related

The main tool will be the language of internal butterflies, and their calculus.

Eventually, I will describe an intrinsic classification for extensions with coefficients in a crossed module in (SH) semi-abelian categories, generalizing a result by Dedecker (1958). .

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Internal groupoids and weak morphisms

The strict monoidal groupoids $D(Y)$ and $\mathrm{AUT}(K)$ are special ones, since their monoidal structure makes them group objects.

They are known as strict categorical groups, or strict 2-groups.

Facts

- *Strict categorical groups are internal groupoids in groups:*

$$\mathbf{Gp}(\mathbf{Gpd}) = \mathbf{Gpd}(\mathbf{Gp})$$

- *Groupoids in groups can be described in terms of crossed modules:*

$$\mathbf{Gpd}(\mathbf{Gp}) \cong \mathbf{XMod}(\mathbf{Gp})$$

This description of groupoids in terms of crossed modules extends to any semi-abelian Category $\mathcal{C}$ [Janelidze 2003], and if the condition (SH) holds, the description simplifies [Martins-Ferreira, Van der Linden 2010].

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Internal crossed modules

In a semi-abelian category $\mathcal{C}$ a notion of internal action is available.
In the case of groups, internal actions correspond to usual group actions:

$$\frac{G_0 \flat G \longrightarrow G}{G_0 \times G \dashrightarrow G}$$

Definition

If $\mathcal{C}$ satisfies (SH), a crossed module $\mathbb{G}$ is a pair

$$G_0 \flat G \xrightarrow{\xi_G} G \xrightarrow{\partial_G} G_0 \ , \quad \text{s.t.}$$

$$\begin{array}{ccc} G_0 \flat G & \xrightarrow{\xi_G} & G \\ {\scriptstyle 1\flat\partial_G}\downarrow & & \downarrow{\scriptstyle \partial_G} \\ G_0 \flat G_0 & \xrightarrow[\chi_{G_0}]{} & G_0 \end{array} \qquad \begin{array}{ccc} G \flat G & \xrightarrow{\chi_G} & G \\ {\scriptstyle \partial_G\flat 1}\downarrow & & \downarrow{\scriptstyle 1_G} \\ G_0 \flat G & \xrightarrow[\xi_G]{} & G \end{array}$$

Definition

- A morphism of (pre)crossed modules $F = (f, f_0) \colon \mathbb{H} \longrightarrow \mathbb{G}$ is a pair of equivariant arrows of $\mathcal{C}$ satisfying:

$$\begin{array}{ccccc} H_0 \flat H & \xrightarrow{\xi_H} & H & \xrightarrow{\partial_H} & H_0 \\ \downarrow{\scriptstyle f_0 \flat f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} \\ G_0 \flat G & \xrightarrow[\xi_G]{} & G & \xrightarrow[\partial_G]{} & G_0 \end{array}$$

- 2-morphism of crossed modules $\alpha \colon F \Rightarrow G \colon \mathbb{H} \to \mathbb{G}$ is an arrow $\alpha \colon H_0 \longrightarrow G \rtimes G_0$ of $\mathcal{C}$ satisfying suitable axioms.

Fact

Crossed modules in $\mathcal{C}$, morphisms and 2-morphisms form a 2-category:

$$\mathbf{XMod}(\mathcal{C})$$

There is another notion of morphism between crossed modules: butterflies. They were introduced for groups [Aldrovandi, Noohi 2009] Internal definition [Abbad, Mantovani, M., Vitale 2013].

Definition

A butterfly $E \colon \mathbb{H} \longrightarrow \mathbb{G}$:

$$\begin{array}{ccccc} E\flat H & \xrightarrow{\sigma\flat 1} & H_0\flat H & \xrightarrow{\xi} & H \\ {\scriptstyle 1\flat\kappa}\downarrow & & & & \downarrow{\scriptstyle \kappa} \\ E\flat E & & \xrightarrow[\chi_E]{} & & E \end{array}$$

$$\begin{array}{ccccc} E\flat G & \xrightarrow{\rho\flat 1} & G_0\flat G & \xrightarrow{\xi} & G \\ {\scriptstyle 1\flat\iota}\downarrow & & & & \downarrow{\scriptstyle \iota} \\ E\flat E & & \xrightarrow[\chi_E]{} & & E \end{array}$$

i. (κ, ρ) is a complex

ii. (ι, σ) is an extension

iii. the two diagrams on the right commute

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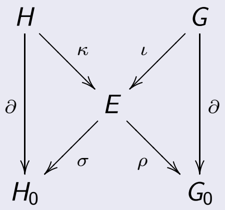

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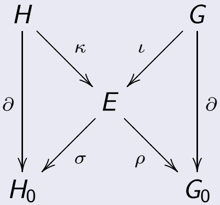

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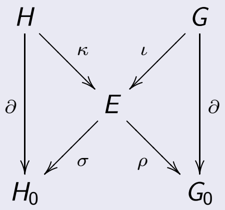

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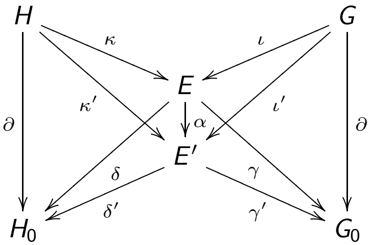
Internal butterflies

A 2-cell $\alpha\colon E \Rightarrow E'$ corresponds to a morphism in $\mathcal{C}$

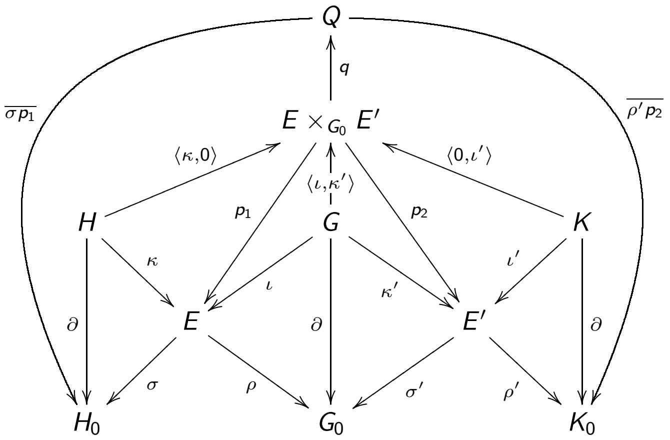
$$\alpha\colon E \to E'$$

s.t. the following diagram commute:



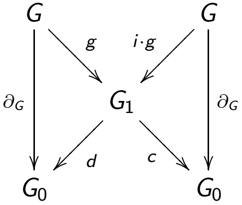
Butterflies composition

For butterflies $E\colon \mathbb{H} \longrightarrow \mathbb{G}$ and $E'\colon \mathbb{G} \longrightarrow \mathbb{K}$. The composite $E' \cdot E\colon \mathbb{H} \longrightarrow \mathbb{K}$ is defined by the following construction:



Identity butterflies

For each crossed module $\mathbb{G} = (\partial_G, \xi_G)$, its identity butterfly is given by the diagram



where $G_1 = G \rtimes G_0$, $g = \ker c$, and $i\colon G_1 \to G_1$ is the inverse map of the associated groupoid structure.

Fact

Crossed modules, butterflies and their 2-cells form bicategory:

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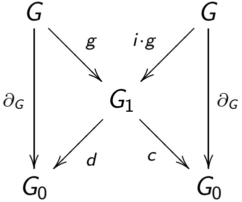

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Proposition (Abbad, Mantovani, M., Vitale 2011)

There exists a canonical embedding

$$\mathcal{B}\colon \mathbf{XMod}(\mathcal{C}) \longrightarrow \mathbf{Bfly}(\mathcal{C})$$

that presents $\mathbf{Bfly}(\mathcal{C})$ *as the bicategory of fractions of* $\mathbf{XMod}(\mathcal{C})$*, w.r.t. internal weak equivalences.*

- *When* $\mathcal{C} = \mathbf{Gp}$ *butterflies describe weak monoidal functors.*
- *Butterflies are certain normalized internal distributors (profunctors).*

Some interesting classes of butterflies

Definition

A butterfly is called *split* if the short exact sequence is split.

Proposition

A butterfly E is split iff it is representable, i.e. $E \cong \mathcal{B}(F)$, for F a strict morphism of crossed modules.

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A butterfly is called *flippable* if both sequences are short exact.

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A butterfly E is flippable iff it is an equivalence in the bicategory $\mathbf{Bfly}(\mathcal{C})$.

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Homotopy invariant for crossed modules

With any crossed module $\mathbb{G} = (\partial_G, \xi_G)$ in $\mathcal{C}$, it is possible to associate two objects (the second being abelian):

$$\pi_0(\mathbb{G}) = \mathrm{Coker}(\partial_G), \qquad \pi_1(\mathbb{G}) = \mathrm{Ker}(\partial_G)$$

and ξ_G induces an action

$$\bar{\xi}_G : \pi_0(\mathbb{G}) \flat \pi_1(\mathbb{G}) \longrightarrow \pi_1(\mathbb{G})$$

that makes $\pi_1(\mathbb{G})$ a $\pi_0(\mathbb{G})$-module.

The assignment extends to a 2-functor

$$\pi_{0,1} : \mathbf{XMod}(\mathcal{C}) \to \mathbf{Mod}(\mathcal{C})$$

The denormalized version of this coincides with Bourn's *global direction*.

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Homotopy invariant for crossed modules

Proposition

The 2-functor $\pi_{0,1}$ extends to butterflies, i.e. there exists a homomorphisms of bicategories (dashed arrows below) that make the following triangle commute

$$\begin{array}{ccc} \mathbf{XMod}(\mathcal{C}) & \xrightarrow{\mathcal{B}} & \mathbf{Bfly}(\mathcal{C}) \\ & \searrow^{\pi_{0,1}} & \downarrow{\scriptstyle \pi_{0,1}} \\ & & \mathbf{Mod}(\mathcal{C}) \end{array}$$

Proof.

1. W.e. in $\mathbf{XMod}(\mathcal{C})$ can be characterized as those morphisms inducing iso on $\pi_{0,1}$
2. $\pi_{0,1}$ sends weak equivalences to equivalences. □

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Extension theory

In the last part of the talk, I will focus on the application of the calculus of butterflies to extension theory.

The approach closely follows Bourn's theory. The use of a normalized version of distributors make it possible to perform the calculations directly with the short exact sequences.

The technique extends the one developed by Noohi for groups.

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Baer sums as butterfly composition

Given $A \xrightarrow{k} X \xrightarrow{f} Y$ and $A \xrightarrow{k'} X' \xrightarrow{f'} Y$, s.e.s of groups inducing the same Y-action α on the abelian kernel A.

$$X \oplus X' = \nabla_A (X \times X') \Delta_Y$$

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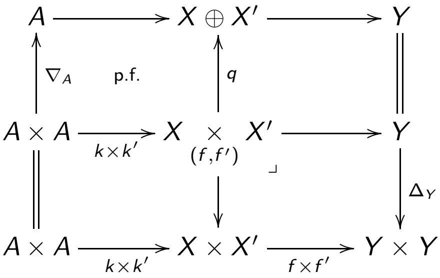

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Baer sums as butterfly composition

Given $A \xrightarrow{k} X \xrightarrow{f} Y$ and $A \xrightarrow{k'} X' \xrightarrow{f'} Y$, short exact sequences of groups inducing the same Y-action ϕ on the abelian kernel A, their Baer sum is

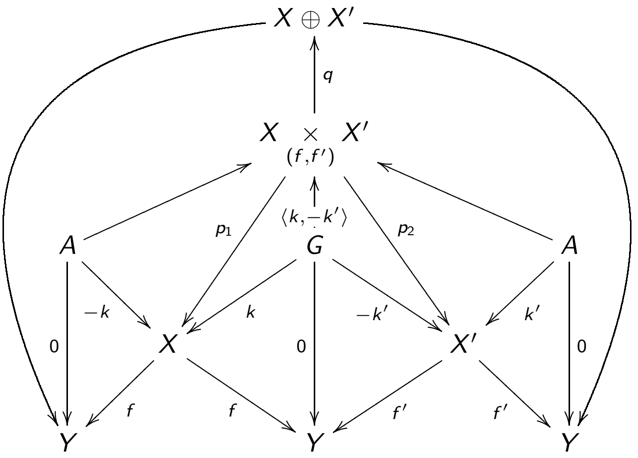
$$X \oplus X' = \nabla_A (X \times X') \Delta_Y$$

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \longrightarrow & X \oplus X' & \longrightarrow & Y \\
\uparrow & & \uparrow{\scriptstyle \nabla_A} & \text{p.f.} & \uparrow{\scriptstyle q} & & \| \\
A & \xrightarrow[\langle 1,-1 \rangle]{} & A \times A & \xrightarrow[k \times k']{} & X \times_{(f,f')} X' & \longrightarrow & Y \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \Delta_Y} \\
 & & A \times A & \xrightarrow[k \times k']{} & X \times X' & \xrightarrow[f \times f']{} & Y \times Y
\end{array}$$

$$q = \mathrm{coker}(\langle k, -k' \rangle)$$

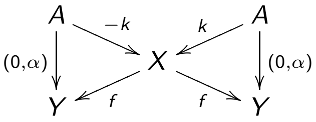
Baer sums as butterfly composition

We can recognize the composition of butterflies:



The assignment

$$A \xrightarrow{k} X \xrightarrow{f} Y \qquad \mapsto$$



defines an embedding $\mathrm{OPEXT}(Y, A, \alpha) \to \mathbf{Bfly}_{eq}(\mathcal{C})\big((0,\alpha),(0,\alpha)\big)$.
Its image

$$\mathbf{Bfly}_1(\mathcal{C})\big((0,\alpha),(0,\alpha)\big)$$

is given by the butterflies X s.t. $\pi_{0,1}(X) = id$.
This is a symm. categorical group, whose classif. group is ($\cong$) $H^2_\alpha(Y, A)$.

Constructions are internal, provided we use internal actions.

Definition

$$H^2_\alpha(Y, A) = \Pi_0\big(\mathbf{Bfly}_1(\mathcal{C})\big((0,\alpha),(0,\alpha)\big)\big)$$

The assignment

$$A \xrightarrow{k} X \xrightarrow{f} Y \quad \mapsto \quad \begin{array}{ccccc} A & \xrightarrow{-k} & & \xleftarrow{k} & A \\ {\scriptstyle (0,\alpha)}\downarrow & & X & & \downarrow{\scriptstyle (0,\alpha)} \\ Y & \xleftarrow{f} & & \xrightarrow{f} & Y \end{array}$$

defines an embedding $\mathrm{OPEXT}(Y, A, \alpha) \to \mathbf{Bfly}_{eq}(\mathcal{C})\big((0,\alpha),(0,\alpha)\big)$.
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Extensions with coeff. in a crossed module

Recall from the introduction, that for groups

$$\mathrm{EXT}(Y,K) \simeq \mathbf{CG}(D(Y),\mathrm{AUT}(K))$$

$$\simeq \quad \mathbf{Bfly}(0_Y,\mathcal{I}_K) \quad \simeq \quad \left\{ \begin{array}{ccccc} 0 & & & & K \\ & \searrow & & \swarrow^{k} & \\ {\scriptstyle 0_Y}\downarrow & & X & & \downarrow{\scriptstyle \mathcal{I}_K} \\ & \swarrow^{f} & & \searrow^{\chi} & \\ Y & & & & \mathrm{Aut}(K) \end{array} \right\}$$

Now,

$$\pi_0(0_Y) = Y \qquad \pi_0(\mathcal{I}_K) = \mathrm{Out}(K)$$

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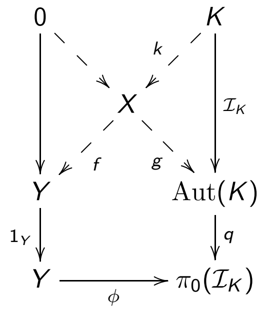
Extension problem

Given $\phi\colon Y \to \pi_0(\mathcal{I}_K)$, classify all the butterflies that fits in the diagram:



The cohomological classification of the extensions inducing a given abstract kernel ϕ becomes an instance of the homotopical classification of weak maps between groupoids (= crossed modules).

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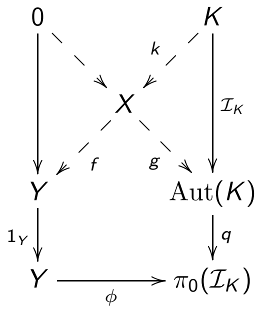

The cohomological classification of the extensions inducing a given abstract kernel ϕ becomes an instance of the homotopical classification of weak maps between groupoids (= crossed modules).

This scheme applies to action representative categories, and, more generally, to the categories where there is some preferred choice for replacing $\mathcal{I}_K$. If this is not the case, a more general problem still arises.

Extension problem

In a (SH) semi-abelian category $\mathcal{C}$, given a crossed module $\mathbb{K} = (\partial_K, \xi_K)$ and a morphism $\phi \colon Y \to \pi_0(\mathbb{K})$, determine the set $\mathrm{OpExt}(Y, \mathbb{K}, \phi)$ of the extensions of Y with coefficients in $\mathbb{K}$ inducing ϕ.

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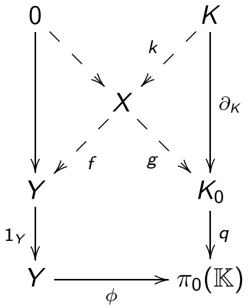


Extensions with coeff. in a crossed module

Theorem

Given the pair

$$\mathbb{K} = (\partial_K, \xi_K) \qquad \phi\colon Y \to \pi_0(\mathbb{K})$$

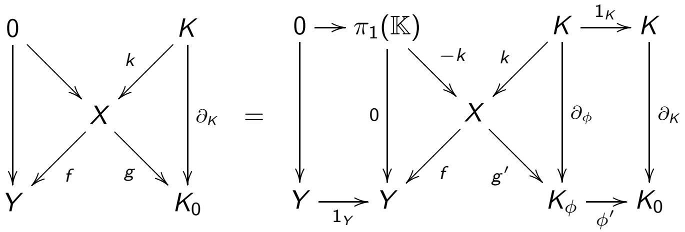
if we denote by

$$\alpha = \phi^*(\overline{\xi}_K) \qquad \partial_\phi = \phi^*(\partial_K)$$

1. *Either* $\mathrm{OpExt}(Y, \mathbb{K}, \phi) = \emptyset$ *or it is a simply transitive* $\mathrm{H}^2_\alpha(Y, \pi_1(\mathbb{K}))$*-set.*
2. $\mathrm{OpExt}(Y, \mathbb{K}, \phi) \neq \emptyset$ *iff* $\mathbf{Bfly}_1((0, \alpha), (\partial_\phi, \xi_\phi)) \neq \emptyset$.

Condition 2 can be interpreted as $\partial_\phi = [0]$ in $H^3_\alpha(Y, \pi_1(\mathbb{K}))$.

More explicitly, one can factor



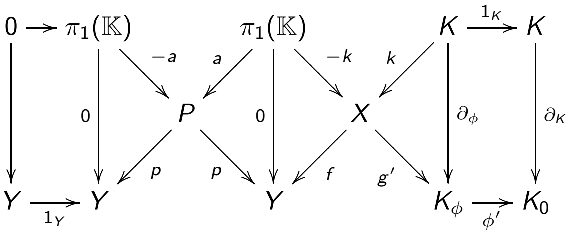
and the action

$$\mathrm{OpExt}(Y, \mathbb{K}, \phi) \times \mathrm{H}^2_\alpha(Y, \pi_1(\mathbb{K})) \to \mathrm{OpExt}(Y, \mathbb{K}, \phi) \; :$$

is just the butterfly composition

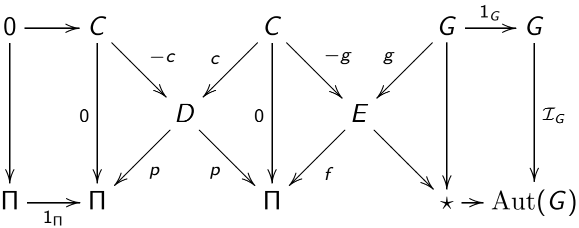
$$\mathbf{Bfly}_1\big((0,\alpha),(\partial_\phi,\xi_\phi)\big) \times \mathbf{Bfly}_1\big((0,\alpha),(0,\alpha)\big) \to \mathbf{Bfly}_1\big((0,\alpha),(\partial_\phi,\xi_\phi)\big)\,.$$

more



The operations of H^2 on Opext may also be defined in invariant terms, without using factor sets. Represent an element of $H^2(\Pi, C)$, according to Thm. 4.1, as an extension D of C by Π with the indicated operators. Let $C \times G$ be the cartesian product of the groups C and G. Define a "codiagonal" map $\nabla\colon C \times G \to G$ by setting $\nabla(c, g) = c + g$; since C is the center of G, this is a homomorphism. The result of operating with D on an extension E in Opext(Π, G, ψ) may then be written as $\nabla(D \times E)\,\Delta_\Pi$. Exactly as in the case of the Baer sum (Ex. 4.7) this does yield an extension of G by Π with the operators ψ;

S. Mac Lane, Homology, IV Cohomology of groups.



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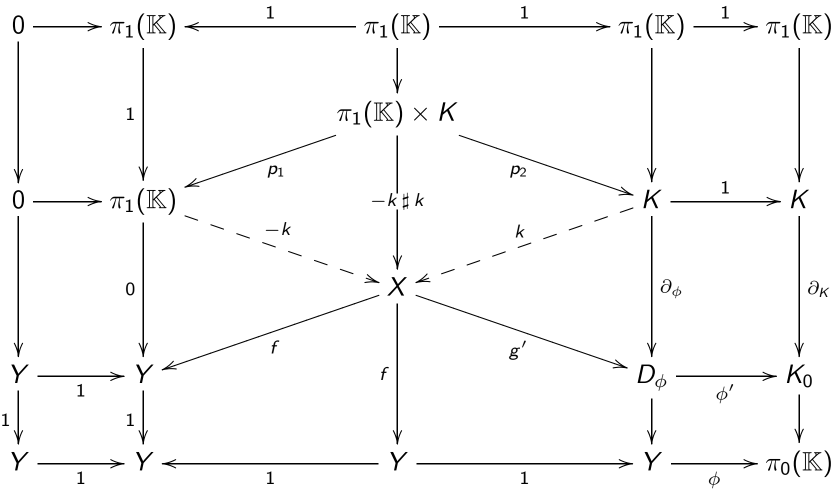
Thank you for your attention!

BUON COMPLEANNO, RUDGER!

Proof



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Dφ
K0
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π0(𝕂)
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Proof

Lemma

Let $\mathcal{B}$ be a groupoid, and x, y be objects of $\mathcal{G}$. Then:

(i) *either $\mathcal{B}(x, y)$ is empty, or arrows composition in $\mathcal{G}$*

$$\mathcal{B}(x, y) \times \mathcal{B}(x, x) \to \mathcal{B}(x, y)$$

defines a simply transitive action $\cdot$ of $\mathcal{G}(x, x)$ on $\mathcal{G}(x, y)$;

(ii) *if $B < \mathcal{B}(x, x)$, and $f \in \mathcal{B}(x, y)$, the action restricts to a simply transitive action*

$$fB \times B \to fB\,, \qquad \text{where } fB = \{f \cdot b \,|\, b \in B\}.$$

Apply to a functor $P \colon \mathcal{B} \to \mathcal{M}$, with $B = \mathcal{B}_{id_x}(x, x)$. back

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