How to get a snail from a butterfly

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Overview

- Intro
- Internal groupoids and weak morphisms
- Internal crossed modules and butterflies
- Kernels of butterflies
- The snake and the snail

Intro

In 1970 Brown shows how to get a six terms exact sequence starting with a fibration of groupoids.

In 2002 Hardie, Kamps and Kieboom show how to get a nine terms exact sequence starting with a fibration of bigroupoids (= weak 2-groupoids).

In 2004 Duskin, Kieboom and Vitale show that in both cases the fibration condition is not needed, provided we use homotopy fibers, instead of strict fibers.

In this talk, we will show how the internalization of a (semi-strict) version of these results is related with the so-called *Snail Lemma* [Vitale 2013].

Categorical Groups

Recall that a categorical group

$$\mathbb{A} = (\mathbb{A}, \otimes, I_A, \dots)$$

is a monoidal groupoid where every object is weakly invertible w.r.t. the tensor product.
A categorical group is *strict* if it is strict as a monoidal category, with strict $\otimes$-inverses.

- Categorical groups with monoidal functors and monoidal natural transformations form the 2-category **CG**
- Strict categorical groups with monoidal functors and monoidal natural transformations form the 2-category $\mathbf{CG}_{mon}$
- Strict categorical groups with strict monoidal functors and strict monoidal natural transformations form the 2-category $\mathbf{CG}_{str}$

All morphisms and 2-morphisms are supposed to be pointed.

Homotopy invariants

With $\mathbb{A}$, it is possible to associate

- the group $\pi_0(\mathbb{A})$ of the connected components of $\mathbb{A}$
- the abelian group $\pi_1(\mathbb{A})$ of the automorphisms of I_A

This process gives rise to two 2-functors:

$$\pi_1 \colon \mathbf{CG} \to \mathbf{Ab}, \qquad \pi_0 \colon \mathbf{CG} \to \mathbf{Gp}.$$

where **Ab** and **Gp** are considered as locally discrete 2-categories.

Remark. π_0 and π_1 are components of a functor $\Pi = (\pi_0, \pi_1)$, and $\pi_1(\mathbb{A})$ has a structure of a $\pi_0(\mathbb{A})$-module.

Kernels in **CG**

Let us consider a pointed morphism of categorical groups, i.e.

$$E\colon \mathbb{A} \to \mathbb{B}, \quad \text{with} \quad E(I_A) = I_B.$$

- the h-kernel $\mathbb{N}_E$ of E is the comma groupoid I_B/E (+ obvious monoidal structure)
- $N_E\colon \mathbb{N}_E \to \mathbb{A}$ is the canonical inclusion of the fiber
- $\nu_E\colon 0 \Rightarrow E \cdot N_E$ is the 2-morphism

$$(A, b\colon I_B \to E(A)) \quad \mapsto \quad b\colon I_B \to E(A).$$

We obtain a 2-categorical kernel sequence (N_E, E, ν_E)

$$\begin{tikzcd}
\mathbb{N}_E \arrow[r, "N_E"'] \arrow[rr, bend left=50, "0"{name=Z}] & \mathbb{A} \arrow[r, "E"'] \arrow[Rightarrow, from=Z, "\nu_E"'] & \mathbb{B}
\end{tikzcd}$$

Kernel sequence

Applying π_0 and π_1 we get two exact sequences of groups and of abelian groups that can be connected in a long exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(\mathbb{N}_E) \arrow[r, "\pi_1(N_E)"] & \pi_1(\mathbb{A}) \arrow[r, "\pi_1(E)"] & \pi_1(\mathbb{B}) \arrow[dll, "\delta"'] \\
 & \pi_0(\mathbb{N}_E) \arrow[r, "\pi_0(N_E)"'] & \pi_0(\mathbb{A}) \arrow[r, "\pi_0(E)"'] & \pi_0(\mathbb{B})
\end{tikzcd}$$

The homomorphism δ sends the *loop-element* $x \colon I_B \to I_B$ of $\pi_1(\mathbb{B})$ to the *class-element* $(I_A, x \colon E(I_A) \to E(I_A))$ of $\pi_0(\mathbb{N}_E)$.

Internal groupoids and weak morphisms

Question

Is it possible to develop a similar theory in an intrinsic setting?

Fact

In the chain

$$\mathbf{CG}_{str} \hookrightarrow \mathbf{CG}_{mon} \hookrightarrow \mathbf{CG}$$

the second inclusion is a bi-equivalence.

A strategy to attack the problem

- Describe intrinsically the 2-category $\mathbf{CG}_{str}$
- Characterize categorically the first inclusion

Intrinsic description of $\mathbf{CG}_{str}$

Fact

A group object in **Cat**, *i.e. a strict categorical group is the same as a category in* **Gp**. *More is true:*

$$\mathbf{CG}_{str} \simeq \mathbf{Gp}(\mathbf{Cat}) \simeq \mathbf{Cat}(\mathbf{Gp}) \simeq \mathbf{Gpd}(\mathbf{Gp})$$

A possible option is to consider a semi-abelian category $\mathcal{C}$, in place of the category **Gp** of groups.

Definition

A category $\mathcal{C}$ is called semi-abelian when it is

- Barr-exact
- pointed
- with finite coproducts
- protomodular

Intrinsic description of $\mathbf{CG}_{str}$

Internal categories in $\mathcal{C}$

$$\begin{tikzcd}
A_1 \times_{A_0} A_1 \arrow[r, "m"] & A_1 \arrow[r, shift left=2, "c"] \arrow[r, shift right=2, "d"'] & A_0 \arrow[l, "e" description]
\end{tikzcd}$$

Internal functors:

$$\begin{tikzcd}
A_1 \arrow[r, "F_1"] \arrow[d, shift right=3, "d"'] \arrow[d, shift left=3, "c"] & B_1 \arrow[d, shift right=3, "d"'] \arrow[d, shift left=3, "c"] \\
A_0 \arrow[r, "F_0"'] \arrow[u] & B_0 \arrow[u]
\end{tikzcd}$$

Internal natural transformations:

$$\begin{tikzcd}
A_1 \arrow[r, shift left=1, "F_1"] \arrow[r, shift right=1, "G_1"'] \arrow[d, shift right=3] \arrow[d, shift left=3] & B_1 \arrow[d, shift right=3] \arrow[d, shift left=3] \\
A_0 \arrow[r, shift left=1, "F_0"] \arrow[r, shift right=1, "G_0"'] \arrow[u] \arrow[ur, "\alpha"] & B_0 \arrow[u]
\end{tikzcd}$$

Since $\mathcal{C}$ Mal'cev, $\mathbf{Cat}(\mathcal{C}) = \mathbf{Gpd}(\mathcal{C})$.

Categorical characterization of $\mathbf{CG}_{str} \hookrightarrow \mathbf{CG}_{mon}$

Motivation: a monoidal functor $F = (F_1, F_0)$ between strict categorical groups is not intrinsic, since F_1 and F_0 are *not* group homomorphisms.

Theorem (Vitale 2010)

The inclusion $\mathbf{CG}_{str} \hookrightarrow \mathbf{CG}_{mon}$ is the bicategory of fractions of $\mathbf{CG}_{str}$, w.r.t. the class of internal weak equivalences

Definition (Pronk 1996)

Let $\mathcal{B}$ be a bicategory and Σ a class of 1-cells in $\mathcal{B}$. The bicategory of fractions of $\mathcal{B}$ with respect to Σ is a homomorphism of bicategories

$$\mathcal{P}_\Sigma \colon \mathcal{B} \to \mathcal{B}[\Sigma^{-1}]$$

universal w.r.t. homomorphisms $\mathcal{F} \colon \mathcal{B} \to \mathcal{A}$ such that $\mathcal{F}(S)$ is an equivalence in $\mathcal{A}$, for every $S \in \Sigma$.

This has been generalized:

Theorem (Mantovani, M., Vitale 2011)

Let $\mathcal{E}$ be Barr-exact. The bicategory of fractions of $\mathbf{Gpd}(\mathcal{E})$ *w.r.t. (internal) weak equivalences can be described by* **fractors**, *i.e. internal distributors*

$$\mathbb{A} \xrightarrow{E} \mathbb{B}$$

whose canonical span representation has the left leg a weak equivalence.

Fractors organize in a bicategory $\mathbf{Fract}(\mathcal{E})$.

Fractors give a notion of weak morphism of internal groupoids in $\mathcal{E}$, equivalent to that of monoidal functors when $\mathcal{E} = \mathbf{Gp}$.

At this point one would expect to develop the theory in the pointed Barr-Exact context... but this is still *work in progress!*

In fact we will restrict to $\mathcal{E} = \mathcal{C}$ S-H semi-abelian.

Definition

A semi-abelian category $\mathcal{C}$ is called S-H semi-abelian if the following condition is satisfied:

Two internal equivalence relations Smith-commute

iff

Their 0-classes Huq-commute

(Martins-Ferreira, Van der Linden 2010)

Two internal equivalence relations Smith-commute

iff

Every reflexive graph with $[\mathrm{Ker}(\mathit{dom}), \mathrm{Ker}(\mathit{cod})] = 0$ *is a groupoid.*

Crossed modules in $\mathcal{C}$

When $\mathcal{C}$ is semi-abelian, groupoids in $\mathcal{C}$ can be described equivalently by internal crossed modules [Janelidze 2003].
When $\mathcal{C}$ is S-H semi-abelian, this description gets much easier to deal with [Martins-Ferreira, Van der Linden 2010].

Definition

Let $\mathcal{C}$ be a semi-abelian category.

- An internal action of an object A_0 on an object A is an algebra $\xi_A \colon A_0 \flat A \to A$ for the monad $A_0 \flat - \colon \mathcal{C} \to \mathcal{C}$, where $A_0 \flat A$ is the kernel of
$$[1,0] \colon A_0 + A \to A_0.$$
- The (canonical) conjugation action of an object A on itself is given by the composition
$$\begin{tikzcd} A \flat A \arrow[r] & A + A \arrow[r, "{[1,1]}"] & A, \end{tikzcd}$$

Crossed modules in $\mathcal{C}$

Definition

- A precrossed module $\mathbb{A}$ is a pair (∂_A, ξ_A), with ξ_A an internal action, such that

$$\begin{tikzcd}
A_0 \flat A \arrow[r, "\xi_A"] \arrow[d, "1\flat\partial_A"'] & A \arrow[d, "\partial_A"] \\
A_0 \flat A_0 \arrow[r, "\chi_{A_0}"'] & A_0
\end{tikzcd}$$

- If $\mathcal{C}$ is S-H semi-abelian, a precrossed module $\mathbb{A}$ is a crossed module when it satisfies the Peiffer identity:

$$\begin{tikzcd}
A \flat A \arrow[r, "\chi_A"] \arrow[d, "\partial_A\flat 1"'] & A \arrow[d, "1_A"] \\
A_0 \flat A \arrow[r, "\xi_A"'] & A
\end{tikzcd}$$

Crossed modules in $\mathcal{C}$

Definition

- A morphism of (pre)crossed modules $F = (f, f_0) \colon \mathbb{A} \longrightarrow \mathbb{B}$ is a pair of equivariant arrows of $\mathcal{C}$ satisfying:

$$\begin{tikzcd}
A_0 \flat A \arrow[r, "\xi_A"] \arrow[d, "f_0 \flat f"'] & A \arrow[r, "\partial_A"] \arrow[d, "f"'] & A_0 \arrow[d, "f_0"] \\
B_0 \flat B \arrow[r, "\xi_B"'] & B \arrow[r, "\partial_B"'] & B_0
\end{tikzcd}$$

- A 2-morphism of crossed modules $\alpha \colon F \Rightarrow G \colon \mathbb{A} \to \mathbb{B}$ is an arrow $\alpha \colon A_0 \longrightarrow B \rtimes B_0$ of $\mathcal{C}$ satisfying suitable axioms.

Crossed modules in $\mathcal{C}$

Theorem (Janelidze 2003)

There is an equivalence of categories

$$\mathbf{XMod}(\mathcal{C}) \simeq \mathbf{Gpd}(\mathcal{C})$$

Exercise (Abbad, Mantovani, M., Vitale 2011)

The equivalence above underlies a bi-equivalence of 2-categories

$$\mathbf{XMod}(\mathcal{C}) \simeq \mathbf{Gpd}(\mathcal{C})$$

Crossed modules in $\mathcal{C}$

We can extend the biequivalence:

$$\begin{tikzcd}
\mathbf{XMod}(\mathcal{C}) \arrow[r, phantom, "\simeq"] \arrow[d] & \mathbf{Gpd}(\mathcal{C}) \arrow[d] \\
?\mathbf{Bfly}(\mathcal{C}) \arrow[r, phantom, "\simeq"] & \mathbf{Fract}(\mathcal{C})
\end{tikzcd}$$

As fractors model weak morphisms of groupoids, there is a notion of weak morphism of crossed modules, that corresponds to fractors under the (bi)equivalence: **butterflies**.

Internal butterflies

Butterflies were introduced by E. Aldrovandi and B. Noohi in [2009] for the category of groups. Here we recall their internal definition [Abbad, Mantovani, M., Vitale 2011]. A butterfly $E\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!\mapsto \mathbb{B}$:

$$\begin{tikzcd}
A \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & B \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
A_0 & & B_0
\end{tikzcd}$$

$$\begin{tikzcd}
E\flat A \arrow[r, "\sigma\flat 1"] \arrow[d, "1\flat\kappa"'] & A_0\flat A \arrow[r, "\xi"] & A \arrow[d, "\kappa"] \\
E\flat E \arrow[rr, "\chi_E"'] & & E
\end{tikzcd}$$

$$\begin{tikzcd}
E\flat B \arrow[r, "\rho\flat 1"] \arrow[d, "1\flat\iota"'] & B_0\flat B \arrow[r, "\xi"] & B \arrow[d, "\iota"] \\
E\flat E \arrow[rr, "\chi_E"'] & & E
\end{tikzcd}$$

i. (κ, ρ) is a complex

ii. (ι, σ) is an extension

iii. iv. the two diagrams on the right commute

Internal butterflies

A 2-cell $\alpha\colon E \Rightarrow E'$ corresponds to a morphism in $\mathcal{C}$

$$\alpha\colon E \to E'$$

s.t. the following diagram commute:

$$\begin{tikzcd}
A \arrow[drr, "\kappa"] \arrow[ddrr, "\kappa'"'] \arrow[ddd, "\partial"'] & & & & B \arrow[dll, "\iota"'] \arrow[ddll, "\iota'"] \arrow[ddd, "\partial"] \\
& & E \arrow[d, "\alpha"] \arrow[ddll, "\delta"] \arrow[ddrr, "\gamma"'] & & \\
& & E' \arrow[dll, "\delta'"] \arrow[drr, "\gamma'"'] & & \\
A_0 & & & & B_0
\end{tikzcd}$$

Crossed modules, butterflies and their 2-cells form a locally groupoidal bicategory **Bfly**$(\mathcal{C})$

The bicategorical structure is inherited from that of fractors, in turn, from that of distributors

Butterflies composition

For butterflies $E\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!+\;\, \mathbb{B}$ and $E'\colon \mathbb{B} \longrightarrow\!\!\!\!\!\!+\;\, \mathbb{C}$. The composite $E'\cdot E\colon \mathbb{A} \longrightarrow\!\!\!\!\!\!+\;\, \mathbb{C}$ is defined by the following construction:

$$\begin{tikzcd}
 & & Q \arrow[ddddll, bend right=60, "\overline{\sigma p_1}"'] \arrow[ddddrr, bend left=60, "\overline{\rho' p_2}"] & & \\
 & & E\times_{B_0}E' \arrow[u, "q"] \arrow[ddl, "p_1"'] \arrow[ddr, "p_2"] & & \\
A \arrow[urr, "\langle\kappa{,}0\rangle"] \arrow[dr, "\kappa"] \arrow[dd, "\partial"'] & & B \arrow[u, "\langle\iota{,}\kappa'\rangle"] \arrow[dl, "\iota"] \arrow[dr, "\kappa'"'] \arrow[dd, "\partial"'] & & C \arrow[ull, "\langle 0{,}\iota'\rangle"'] \arrow[dl, "\iota'"'] \arrow[dd, "\partial"] \\
 & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & & E' \arrow[dl, "\sigma'"] \arrow[dr, "\rho'"'] & \\
A_0 & & B_0 & & C_0
\end{tikzcd}$$

It clearly extends to 2 morphisms

Identity butterflies

For each crossed module $\mathbb{B} = (\partial_B, \xi_B)$, its identity butterfly is given by the diagram

$$\begin{tikzcd}
B \arrow[dd, "\partial_B"'] \arrow[dr, "b"] & & B \arrow[dl, "i \cdot b"'] \arrow[dd, "\partial_B"] \\
& B_1 \arrow[dl, "d"] \arrow[dr, "c"'] & \\
B_0 & & B_0
\end{tikzcd}$$

where $B_1 = B \rtimes B_0$ and $i \colon B_1 \to B_1$ is the inverse map of the associated groupoid structure.

Reduced left composition

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[dd, "\partial"'] & H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \\
& & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
Y_0 \arrow[r, "f_0"'] & H_0 & & G_0
\end{tikzcd}$$

Reduced left composition

$$\begin{tikzcd}
Y \arrow[rr, "f"] \arrow[dd, "\partial"'] \arrow[dr, dotted] & & H \arrow[drr, "\kappa"] \arrow[dd, dotted] & & G \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \arrow[dlll, dotted] \\
 & Y \times_{H_0} E \arrow[rrr] \arrow[dl] \arrow[dr, phantom, "p.b."] & & & E \arrow[dll, "\sigma"] \arrow[d, "\rho"'] \\
Y_0 \arrow[rr, "f_0"'] & & H_0 & & G_0
\end{tikzcd}$$

Reduced left composition

$$\begin{tikzcd}
Y \arrow[dd, "\partial"'] \arrow[dr] \arrow[rr, dotted] & & {} \arrow[dd, dotted] \arrow[dr, dotted] & & G \arrow[dd, "\partial"] \arrow[dl, "\iota"'] \arrow[dlll] \\
& Y \times_{H_0} E \arrow[rr] \arrow[dl] \arrow[dr, phantom, "p.b."] & & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
Y_0 \arrow[rr, "f_0"'] & & H_0 & & G_0
\end{tikzcd}$$

Reduced left composition

$$\begin{tikzcd}
Y \arrow[dd, "\partial"'] \arrow[dr] \arrow[rr, dotted] & & {} \arrow[dr, dotted] & & G \arrow[dlll] \arrow[dl, dotted] \arrow[dd, "\partial"] \\
& Y \times_{H_0} E \arrow[dl] \arrow[rr, dotted] \arrow[drrr] & & {} \arrow[dl, dotted] \arrow[dr, dotted] & \\
Y_0 \arrow[rr, dotted] & & {} & & G_0
\end{tikzcd}$$

Reduced left composition

$$\begin{tikzcd}
Y \arrow[dd, "\partial"'] \arrow[dr] & & G \arrow[dl] \arrow[dd, "\partial"] \\
& Y \times_{H_0} E \arrow[dl] \arrow[dr] & \\
Y_0 & & G_0
\end{tikzcd}$$

Reduced left composition

$$\begin{tikzcd}
Y \arrow[dd, "\partial"'] \arrow[drr, "\langle \partial{,}\kappa\cdot f\rangle"] & & & & G \arrow[dll, "\langle 0{,}\iota\rangle"'] \arrow[dd, "\partial"] \\
& & Y \times_{H_0} E \arrow[dll, "p_2"] \arrow[drr, "\rho\cdot p_1"'] & & \\
Y_0 & & & & G_0
\end{tikzcd}$$

Reduced right composition

$$\begin{tikzcd}
H \arrow[dr, "\kappa"] \arrow[dd, "\partial"'] & & G \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \arrow[r, "f"] & X \arrow[dd, "\partial"] \\
& E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & & \\
H_0 & & G_0 \arrow[r, "f_0"'] & X_0
\end{tikzcd}$$

Reduced right composition

$$\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dl, "\iota"'] \arrow[rr, "f"] \arrow[dd, dotted] \arrow[dr, phantom, "\text{p.f.}"] & & X \arrow[dl] \arrow[dd, "\partial"] \\
& E \arrow[rr] \arrow[dl, "\sigma"'] \arrow[dr, "\rho"] & & E \rtimes^{G} X \arrow[dr] \arrow[dlll, dotted] & \\
H_0 & & G_0 \arrow[rr, "f_0"'] & & X_0
\end{tikzcd}$$

Reduced right composition

$$\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dl] \arrow[rr, "f"] \arrow[dd, dotted] \arrow[dr, phantom, "\text{p.f.}"] & & X \arrow[dl] \arrow[dd, "\partial"] \\
& E \arrow[rr] \arrow[dl] \arrow[dr, dotted] & & E \rtimes^{G} X \arrow[dr] \arrow[dlll] & \\
H_0 & & {} \arrow[rr, dotted] & & X_0
\end{tikzcd}$$

Reduced right composition

$$\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[rrrd] \arrow[rd, dotted] & & {} \arrow[rr, dotted] \arrow[ld, dotted] & & X \arrow[ld] \arrow[dd, "\partial"] \\
& {} \arrow[rr, dotted] \arrow[ld, dotted] \arrow[rd, dotted] & & E \rtimes^{G} X \arrow[llld] \arrow[rd] & \\
H_0 & & {} \arrow[rr, dotted] & & X_0
\end{tikzcd}$$

Reduced right composition

$$\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr] & & X \arrow[dl] \arrow[dd, "\partial"] \\
& E \rtimes^{G} X \arrow[dl] \arrow[dr] & \\
H_0 & & X_0
\end{tikzcd}$$

Reduced right composition

$$\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr] & & X \arrow[dl] \arrow[dd, "\partial"] \\
& E \rtimes^{G} X \arrow[dl] \arrow[dr] & \\
H_0 & & X_0
\end{tikzcd}$$

One description of $\mathcal{B}$

$$\mathcal{B}\colon \mathbf{XMod}(\mathcal{C}) \to \mathbf{Bfly}(\mathcal{C})$$

For $F\colon \mathbb{A} \to \mathbb{B}$ (strict) morphism of crossed module

$$F \mapsto \mathcal{B}(F) = F \circ_r Id_{\mathbb{A}}$$

The result is a split-butterfly (the short exact sequence is split).

Fact

All split butterflies arise this way.

Homotopy invariant for crossed modules

Definition

For a crossed module $\mathbb{A} = (\partial_A, \xi_A)$ in $\mathcal{C}$:

$$\pi_0(\mathbb{A}) = \mathrm{Coker}(\partial_A)\,, \qquad \pi_1(\mathbb{A}) = \mathrm{Ker}(\partial_A)\,.$$

Fact

π_1 and π_0 yield two 2-functors:

$$\pi_1 : \mathbf{XMod}(\mathcal{C}) \longrightarrow \mathbf{Ab}(\mathcal{C}) \;, \qquad \pi_0 : \mathbf{XMod}(\mathcal{C}) \longrightarrow \mathcal{C} \;,$$

Definition

A morphism of crossed modules is called *weak equivalence* if it determines a weak equivalence between the corresponding groupoids. Weak equivalences can be characterized as those morphisms of crossed modules inducing isomorphisms on π_0 and π_1.

Homotopy invariant for crossed modules

Proposition

The 2-functors π_0 and π_1 extend to butterflies, i.e. there exist two homomorphisms of bicategories (dashed arrows below) that make the two following triangles commute

$$\begin{tikzcd}
\mathbf{XMod}(\mathcal{C}) \arrow[r, "\mathcal{B}"] \arrow[dr, "\pi_1"'] & \mathbf{Bfly}(\mathcal{C}) \arrow[d, dashed, "\pi_1"] \\
 & \mathbf{Ab}(\mathcal{C})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{XMod}(\mathcal{C}) \arrow[r, "\mathcal{B}"] \arrow[dr, "\pi_0"'] & \mathbf{Bfly}(\mathcal{C}) \arrow[d, dashed, "\pi_0"] \\
 & \mathcal{C}
\end{tikzcd}$$

Proof.

- $\mathcal{B}$ presents $\mathbf{Bfly}(\mathcal{C})$ as the bicategory of fractions of $\mathbf{XMod}(\mathcal{C})$ w.r.t. weak equivalences
- both π_0 and π_1 send weak equivalences in equivalences

$\square$

Explicit description of π_0 and π_1

Distributors can be represented by spans of functors.
In the language of crossed modules, butterflies can be represented by spans of crossed modules morphisms:

$$\begin{tikzcd}
& A\times B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "\kappa\sharp\iota"] & \\
A \arrow[dr, dashed, "\kappa"] \arrow[dd, "\partial_A"'] & & B \arrow[dl, dashed, "\iota"'] \arrow[dd, "\partial_B"] \\
& E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
A_0 & & B_0
\end{tikzcd}$$

1. $\kappa\sharp\iota$ is the cooperator of κ and ι
2. (p_1, σ) is a weak equivalence

Proposition

For the butterfly $E\colon \mathbb{A} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{B}$

$$\pi_0(E) = \pi_0(p_2, \rho) \cdot (\pi_0(p_1, \sigma))^{-1}\,, \qquad \pi_1(E) = \pi_1(p_2, \rho) \cdot (\pi_1(p_1, \sigma))^{-1}\,.$$

What kernel for butterflies?

Definition

The bikernel of a butterfly E is the bipullback over the cospan

$$\begin{tikzcd} & \mathbb{O} \arrow[d] \\ \mathbb{A} \arrow[r, "\shortmid" marking] & \mathbb{B} \end{tikzcd}$$

where $\mathbb{O}$ is the trivial crossed module.

Fact

Bipullbacks in $\mathcal{B}(\mathcal{C})$ are equivalent to bi-isocomma squares

Fact

Isocomma squares (defined up to iso) are instances of bi-isocomma squares (defined up to equivalences).

What kernel for butterflies?

Definition

For a butterfly $E\colon \mathbb{A} \xrightarrow{E} \mathbb{B}$, the triple $(\mathbb{N}, N, \nu)$ is a strict isocomma kernel of E if it satisfies the following universal property:
1. For any other $(\mathbb{L}, M, \mu)$, there exists a unique X s. t.

$$\begin{tikzcd}
\mathbb{N} \arrow[dr, "N"] \arrow[drr, bend left=40, "0_{N,B}"] & {} \arrow[d, phantom, "\Downarrow \nu"] & \\
 & \mathbb{A} \arrow[r, "E", "|" marking] & \mathbb{B} \\
\mathbb{L} \arrow[uu, "X", "|" marking] \arrow[ur, "M"', "/" marking] \arrow[urr, bend right=40, "0_{L,B}"'] & {} \arrow[u, phantom, "\Uparrow \mu"] &
\end{tikzcd}$$

$$N \cdot X = M$$
$$\alpha(\nu \cdot X) = \mu\zeta$$

where α is the associator, and ζ is the canonical comparison between zero-morphisms.

What kernel for butterflies?

2. For any pair of morphisms $X, Y \colon \mathbb{L} \longrightarrow \mathbb{N}$, and for any 2-morphism $\varphi \colon N \cdot X \Longrightarrow N \cdot Y$ such that:

$$\begin{tikzcd}
0_{N,B} \cdot X \arrow[rrr, Rightarrow, "\zeta"] \arrow[d, Rightarrow, "\nu \cdot X"'] & & & 0_{N,B} \cdot Y \arrow[d, Rightarrow, "\nu \cdot Y"] \\
(E \cdot N) \cdot X \arrow[r, Rightarrow, "\alpha"'] & E \cdot (N \cdot X) \arrow[r, Rightarrow, "E \cdot \varphi"'] & E \cdot (N \cdot Y) \arrow[r, Rightarrow, "\alpha^{-1}"'] & (E \cdot N) \cdot Y
\end{tikzcd}$$

there exists a unique $\psi \colon X \Longrightarrow Y$ such that

$$N \cdot \psi = \varphi \, .$$

Kernels: preliminary constructions

Null morphisms of crossed modules via butterflies: $\mathcal{B}(0_{A,B})$

$$\begin{tikzcd}
A \arrow[r, "0"] \arrow[d, "\partial"'] & B \arrow[d, "\partial"'] \arrow[dr, "b"] & & B \arrow[dl, "ib"'] \arrow[d, "\partial"] & & A \arrow[d, "\partial"'] \arrow[dr, "\langle\partial{,}0\rangle"] & & B \arrow[dl, "\langle 0{,}1\rangle"'] \arrow[d, "\partial"] \\
 & & B_1 \arrow[dl, "d"] \arrow[dr, "c"'] & & = & & A_0 \times B \arrow[dl, "\pi_1"] \arrow[dr, "\partial\pi_2"'] & \\
A_0 \arrow[r, "0"'] & B_0 & & B_0 & & A_0 & & B_0
\end{tikzcd}$$

Kernels: preliminary constructions

Strategy: transpose the construction of the classical h-kernel for crossed modules in terms of butterflies and next we shall prove that it satisfies the corresponding universal property. (see [Aldrovandi, Noohi 2009] for the case of groups)

Kernels: preliminary constructions

Given a morphism of crossed modules $F = (f, f_0)\colon \mathbb{A} \to \mathbb{B}$, consider the pullback of ∂_B along f_0, and the comparison $\overline{\partial} = \langle \partial, f \rangle$:

$$\begin{tikzcd}
A \arrow[rrd, bend left, "f"] \arrow[rd, "\overline{\partial}"] \arrow[rdd, bend right, "\partial"'] & & \\
& A_0 \times_{B_0} B \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \arrow[d, "\partial"] \\
& A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

Then p_1 is a crossed modules and the diagram above can be interpreted as the (constant on objects / fully faithful) factorization of F:
$(f, f_0) = (p_2, f_0) \cdot (\overline{\partial}, 1_{A_0})$.

Kernels: preliminary constructions

Lemma

Also $\overline{\partial}$ *is a crossed module, with action* $p_1^*(\xi_A)$*, and the pair* $N = (1_A, p_1)$ *is a morphism of crossed modules:*

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "\overline{\partial}"'] & A \arrow[r, "f"] \arrow[d, "\partial"] & B \arrow[d, "\partial"] \\
A_0 \times_{B_0} B \arrow[r, "p_1"'] \arrow[urr, dashed] & A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

As we will see, this is an instance of the bikernel of the strict morphism $F = (f, f_0)$ and the (missing) projection

$$p_2 \colon A_0 \times_{B_0} B \to B$$

is precisely the null map

$$\nu \colon E \Rightarrow E \cdot N$$

The kernel of a butterfly

For a butterfly $E\colon \mathbb{A} \mathrel{-\!\!\!+\!\!\!\!\rightarrow} \mathbb{B}$. , we take:

- $n = \ker \rho$
- $\overline{\kappa}$ such that $n \cdot \overline{\kappa} = \kappa$

We obtain

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "N"'] \arrow[rr, bend left=60, "0_{N,B}", ""{name=U, below}] & \mathbb{A} \arrow[r, "E"', "|" marking] & \mathbb{B}
\arrow[Rightarrow, from=U, to=1-2, "\nu"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[rr, "1"] \arrow[dd, "\overline{\kappa}"'] & & A \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & B \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \\
& & & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
N_0 \arrow[r, "n"'] & E \arrow[r, "\sigma"'] & A_0 & & B_0
\end{tikzcd}$$

- a crossed module $\mathbb{N} = \overline{\kappa}$
- a morphism $N = (1_A, \sigma \cdot n)\colon \mathbb{N} \to \mathbb{A}$

The kernel of a butterfly

Lemma

With hypothesis and notation as above, we get a canonical isomorphism

$$\nu\colon E \cdot N \Rightarrow 0_{N,B}$$

Proposition

The construction above gives a strict isocomma kernel of E, hence an instance of a bikernel of E.

A possible explanation of the construction

In the 2-category $\mathbf{CG}_{mon}$, the span associated to a monoidal functor E can be computed as an iso-comma object:

$$\begin{tikzcd}
\mathbb{E} \arrow[r, "R"] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "\Rightarrow"] & \mathbb{B} \arrow[d, "1"] \\
\mathbb{A} \arrow[r, "|" marking, "E"'] & \mathbb{B}
\end{tikzcd}$$

By the pasting property of isocomma squares and pullbacks one gets

$$\begin{tikzcd}
\mathbb{N} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{O} \arrow[d] \\
\mathbb{E} \arrow[r, "R"] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "\Rightarrow"] & \mathbb{B} \arrow[d, "1"] \\
\mathbb{A} \arrow[r, "|" marking, "E"'] & \mathbb{B}
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{E} \arrow[r] \arrow[d, "N"'] \arrow[dr, phantom, "\Rightarrow"] & \mathbb{O} \arrow[d] \\
\mathbb{A} \arrow[r, "|" marking, "E"'] & \mathbb{B}
\end{tikzcd}$$

The fiber sequence and the snail lemma

Now we can come back to our initial problem.

Proposition

From a kernel diagram

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "N"'] \arrow[rr, bend left, "0", ""{name=U, below}] & \mathbb{A} \arrow[r, "E"', "|" marking] & \mathbb{B}
\arrow[from=U, to=1-2, Rightarrow, "\nu"]
\end{tikzcd}$$

one obtains the long exact sequence:

$$0 \to \pi_1(\mathbb{K}) \to \pi_1(\mathbb{A}) \to \pi_1(\mathbb{B}) \xrightarrow{\sigma} \pi_0(\mathbb{K}) \to \pi_0(\mathbb{A}) \to \pi_0(\mathbb{B})$$

The fiber sequence: proof

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(\mathbb{N}) \arrow[r] \arrow[dd] & \pi_1(\kappa\sharp\iota) \arrow[r, "\sim"] \arrow[dd, "1"] & \pi_1(\mathbb{A}) \arrow[dd] & \pi_1(\kappa\sharp\iota) \arrow[l, "\sim"'] \arrow[r] \arrow[d] & \pi_1(\mathbb{B}) \arrow[dd] \\
 & & & & A \times B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "\kappa\sharp\iota"] & \\
 & A \arrow[r, "{\langle 1,0 \rangle}"] \arrow[dd, "\overline{\kappa}"'] & A \times B \arrow[r, "p_1"] \arrow[dd, "\kappa\sharp\iota"] & A \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & B \arrow[dd, "\partial"] \arrow[dl, "\iota"'] \\
 & & & & E \arrow[dl, "\sigma"'] \arrow[dr, "\rho"] \arrow[dd] & \\
 & N_0 \arrow[r, "n"'] \arrow[d] & E \arrow[r, "\sigma"'] \arrow[d] & A_0 \arrow[d] & & B_0 \arrow[d] \\
 & \pi_0(\mathbb{N}) \arrow[r] & \pi_0(\kappa\sharp\iota) \arrow[r, "\sim"'] & \pi_0(\mathbb{A}) & \pi_0(\kappa\sharp\iota) \arrow[l, "\sim"] \arrow[r] & \pi_0(\mathbb{B})
\end{tikzcd}$$

The fiber sequence: proof

Thus we obtain the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(\mathbb{N}) \arrow[r, "\pi_1(\langle 0{,}1\rangle{,}n)"] & \pi_1(\kappa\sharp\iota) \arrow[r, "\pi_1(p_2{,}\rho)"] & \pi_1(\mathbb{B}) \arrow[dll, "\delta"'] \\
 & \pi_0(\mathbb{N}) \arrow[r, "\pi_0(\langle 0{,}1\rangle{,}n)"'] & \pi_0(\kappa\sharp\iota) \arrow[r, "\pi_0(p_2{,}\rho)"'] & \pi_0(\mathbb{A})
\end{tikzcd}$$

After inserting the compositions

$$\pi_1(p_1,\sigma)^{-1}\cdot\pi_1(p_1,\sigma) \qquad \text{and} \qquad \pi_0(p_1,\sigma)^{-1}\cdot\pi_0(p_1,\sigma)\,,$$

we obtain the desired long exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(\mathbb{N}) \arrow[r, "\pi_1(N)"] & \pi_1(\mathbb{A}) \arrow[r, "\pi_1(E)"] & \pi_1(\mathbb{B}) \arrow[dll, "\delta"'] \\
 & \pi_0(\mathbb{N}) \arrow[r, "\pi_0(N)"'] & \pi_0(\mathbb{A}) \arrow[r, "\pi_0(E)"'] & \pi_0(\mathbb{B})
\end{tikzcd}$$

The snail lemma for crossed modules

Corollary

With any crossed modules (strict) morphism $(f, f_0)\colon \mathbb{A} \longrightarrow \mathbb{B}$ *it is possible to associate a long exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(\overline{\partial}) \arrow[r] & \pi_1(\mathbb{A}) \arrow[r, "\pi_1(F)"] & \pi_1(\mathbb{B}) \arrow[dll, "\delta"'] \\
& \pi_0(\overline{\partial}) \arrow[r] & \pi_0(\mathbb{A}) \arrow[r, "\pi_0(F)"'] & \pi_0(\mathbb{B})
\end{tikzcd}$$

Proof.

It suffices to apply the previous result to the split butterfly $E_F = \mathcal{B}(F)$. □

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Thank you for your attention!