

Kernel sequence

Applying π_0 and π_1 we get two exact sequences of groups and of abelian groups that can be connected in a long exact sequence:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \pi_1(\mathbb{N}_E) & \xrightarrow{\pi_1(N_E)} & \pi_1(\mathbb{A}) & \xrightarrow{\pi_1(E)} & \pi_1(\mathbb{B}) \\
 & & & & & \searrow \delta & \\
 & & \pi_0(\mathbb{N}_E) & \xrightarrow{\pi_0(N_E)} & \pi_0(\mathbb{A}) & \xrightarrow{\pi_0(E)} & \pi_0(\mathbb{B})
 \end{array}$$

The homomorphism δ sends the *loop-element* $x: I_B \rightarrow I_B$ of $\pi_1(\mathbb{B})$ to the *class-element* $(I_A, x: E(I_A) \rightarrow E(I_A))$ of $\pi_0(\mathbb{N}_E)$.

Intrinsic description of \mathbf{CG}_{str}

Internal categories in \mathcal{C}

$$A_1 \times_{A_0} A_1 \xrightarrow{m} A_1 \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{e} \\ \xrightarrow{d} \end{array} A_0$$

Internal functors:

$$\begin{array}{ccc} A_1 & \xrightarrow{F_1} & B_1 \\ \downarrow d \uparrow c & & \downarrow d \uparrow c \\ A_0 & \xrightarrow{F_0} & B_0 \end{array}$$

Internal natural transformations:

$$\begin{array}{ccc} A_1 & \begin{array}{c} \xrightarrow{F_1} \\ \xrightarrow{G_1} \end{array} & B_1 \\ \downarrow \uparrow & \nearrow \alpha & \downarrow \uparrow \\ A_0 & \begin{array}{c} \xrightarrow{F_0} \\ \xrightarrow{G_0} \end{array} & B_0 \end{array}$$

Since \mathcal{C} Mal'cev, $\mathbf{Cat}(\mathcal{C}) = \mathbf{Gpd}(\mathcal{C})$.

Crossed modules in \mathcal{C}

Theorem (Janelidze 2003)

There is an equivalence of categories

$$\mathbf{XMod}(\mathcal{C}) \simeq \mathbf{Gpd}(\mathcal{C})$$

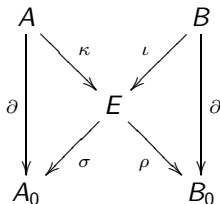
Exercise (Abbad, Mantovani, M., Vitale 2011)

The equivalence above underlies a bi-equivalence of 2-categories

$$\mathbf{XMod}(\mathcal{C}) \simeq \mathbf{Gpd}(\mathcal{C})$$

Internal butterflies

Butterflies were introduced by E. Aldrovandi and B. Noohi in [2009] for the category of groups. Here we recall their internal definition [Abbad, Mantovani, M., Vitale 2011]. A butterfly $E: \mathbb{A} \dashrightarrow \mathbb{B}$:



$$\begin{array}{ccc}
 E \flat A & \xrightarrow{\sigma \flat 1} & A_0 \flat A & \xrightarrow{\xi} & A \\
 1 \flat \kappa \downarrow & & & & \downarrow \kappa \\
 E \flat E & \xrightarrow{\chi_E} & & & E \\
 \\
 E \flat B & \xrightarrow{\rho \flat 1} & B_0 \flat B & \xrightarrow{\xi} & B \\
 1 \flat \iota \downarrow & & & & \downarrow \iota \\
 E \flat E & \xrightarrow{\chi_E} & & & E
 \end{array}$$

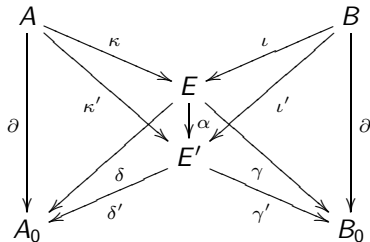
- i. (κ, ρ) is a complex
- ii. (ι, σ) is an extension
- iii. iv. the two diagrams on the right commute

Internal butterflies

A 2-cell $\alpha: E \Rightarrow E'$ corresponds to a morphism in \mathcal{C}

$$\alpha: E \rightarrow E'$$

s.t. the following diagram commute:

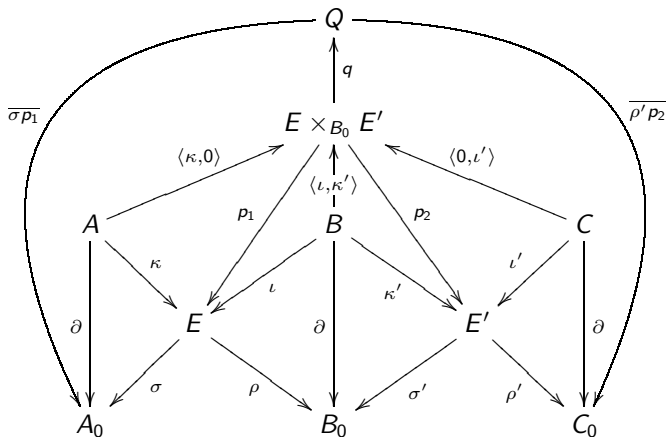


*Crossed modules, butterflies and their 2-cells form a locally groupoidal bicategory **Bfly**(\mathcal{C})*

The bicategorical structure is inherited from that of factors, in turn, from that of distributors

Butterflies composition

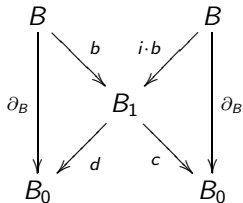
For butterflies $E: \mathbb{A} \twoheadrightarrow \mathbb{B}$ and $E': \mathbb{B} \twoheadrightarrow \mathbb{C}$. The composite $E' \cdot E: \mathbb{A} \twoheadrightarrow \mathbb{C}$ is defined by the following construction:



It clearly extends to 2-morphisms

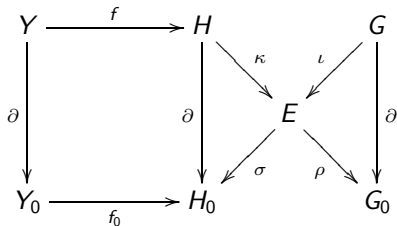
Identity butterflies

For each crossed module $\mathbb{B} = (\partial_B, \xi_B)$, its identity butterfly is given by the diagram

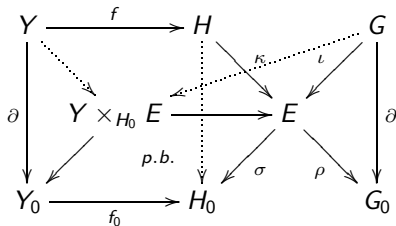


where $B_1 = B \rtimes B_0$ and $i: B_1 \rightarrow B_1$ is the inverse map of the associated groupoid structure.

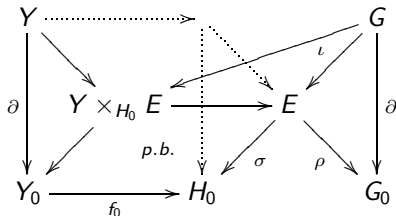
Reduced left composition



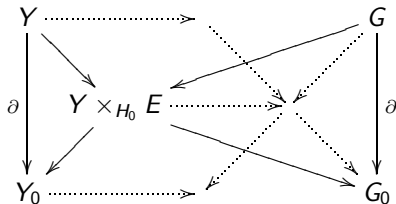
Reduced left composition



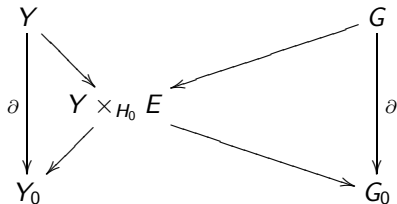
Reduced left composition



Reduced left composition



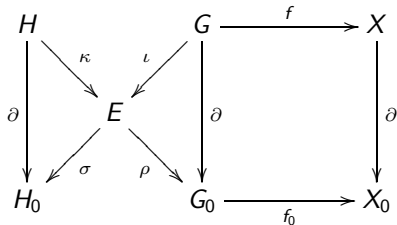
Reduced left composition



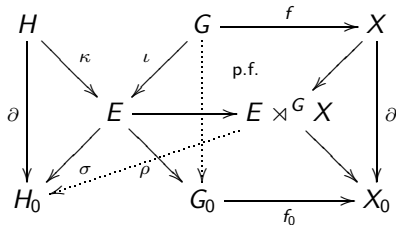
Reduced left composition

$$\begin{array}{ccccc}
 Y & & & & G \\
 \downarrow \partial & \searrow \langle \partial, \kappa \cdot f \rangle & & \swarrow \langle 0, \iota \rangle & \downarrow \partial \\
 & & Y \times_{H_0} E & & \\
 & \swarrow p_2 & & \searrow \rho \cdot p_1 & \\
 Y_0 & & & & G_0
 \end{array}$$

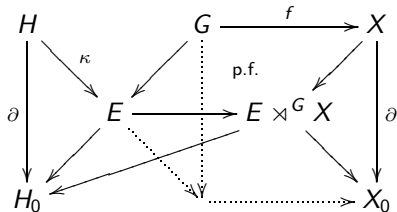
Reduced right composition



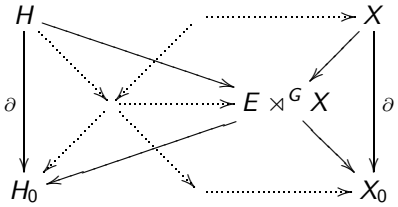
Reduced right composition



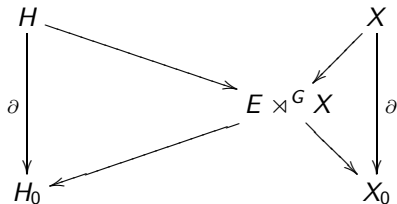
Reduced right composition



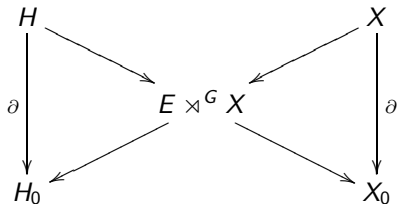
Reduced right composition



Reduced right composition



Reduced right composition



One description of \mathcal{B}

$$\mathcal{B}: \mathbf{XMod}(\mathcal{C}) \rightarrow \mathbf{Bfly}(\mathcal{C})$$

For $F: \mathbb{A} \rightarrow \mathbb{B}$ (strict) morphism of crossed module

$$F \mapsto \mathcal{B}(F) = F \circ_r Id_{\mathbb{A}}$$

The result is a split-butterfly (the short exact sequence is split).

Fact

All split butterflies arise this way.

Homotopy invariant for crossed modules

Definition

For a crossed module $\mathbb{A} = (\partial_A, \xi_A)$ in \mathcal{C} :

$$\pi_0(\mathbb{A}) = \text{Coker}(\partial_A), \quad \pi_1(\mathbb{A}) = \text{Ker}(\partial_A).$$

Fact

π_1 and π_0 yield two 2-functors:

$$\pi_1: \mathbf{XMod}(\mathcal{C}) \longrightarrow \mathbf{Ab}(\mathcal{C}), \quad \pi_0: \mathbf{XMod}(\mathcal{C}) \longrightarrow \mathcal{C},$$

Definition

A morphism of crossed modules is called *weak equivalence* if it determines a weak equivalence between the corresponding groupoids. Weak equivalences can be characterized as those morphisms of crossed modules inducing isomorphisms on π_0 and π_1 .

Homotopy invariant for crossed modules

Proposition

The 2-functors π_0 and π_1 extend to butterflies, i.e. there exist two homomorphisms of bicategories (dashed arrows below) that make the two following triangles commute

$$\begin{array}{ccc}
 \mathbf{XMod}(\mathcal{C}) & \xrightarrow{\mathcal{B}} & \mathbf{Bfly}(\mathcal{C}) \\
 \searrow \pi_1 & & \downarrow \pi_1 \\
 & & \mathbf{Ab}(\mathcal{C})
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbf{XMod}(\mathcal{C}) & \xrightarrow{\mathcal{B}} & \mathbf{Bfly}(\mathcal{C}) \\
 \searrow \pi_0 & & \downarrow \pi_0 \\
 & & \mathcal{C}
 \end{array}$$

Proof.

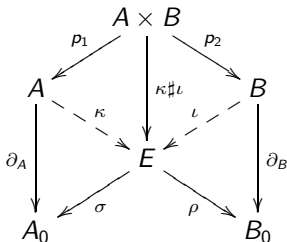
- \mathcal{B} presents $\mathbf{Bfly}(\mathcal{C})$ as the bicategory of fractions of $\mathbf{XMod}(\mathcal{C})$ w.r.t. weak equivalences
- both π_0 and π_1 send weak equivalences in equivalences



Explicit description of π_0 and π_1

Distributors can be represented by spans of functors.

In the language of crossed modules, butterflies can be represented by spans of crossed modules morphisms:



1. $\kappa \sharp l$ is the cooperator of κ and l
2. (p_1, σ) is a weak equivalence

Proposition

For the butterfly $E: \mathbb{A} \twoheadrightarrow \mathbb{B}$

$$\pi_0(E) = \pi_0(p_2, \rho) \cdot (\pi_0(p_1, \sigma))^{-1},$$

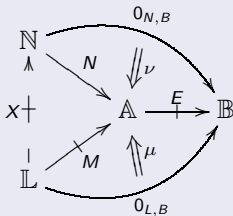
$$\pi_1(E) = \pi_1(p_2, \rho) \cdot (\pi_1(p_1, \sigma))^{-1}.$$

What kernel for butterflies?

Definition

For a butterfly $E: \mathbb{A} \xrightarrow{E} \mathbb{B}$, the triple (\mathbb{N}, N, ν) is a strict isocomma kernel of E if it satisfies the following universal property:

1. For any other (\mathbb{L}, M, μ) , there exists a unique X s. t.



$$N \cdot X = M$$

$$\alpha(\nu \cdot X) = \mu \zeta$$

where α is the associator, and ζ is the canonical comparison between zero-morphisms.

What kernel for butterflies?

2. For any pair of morphisms $X, Y: \mathbb{L} \longrightarrow \mathbb{N}$, and for any 2-morphism $\varphi: N \cdot X \Longrightarrow N \cdot Y$ such that:

$$\begin{array}{ccc}
 0_{N,B} \cdot X & \xrightleftharpoons{\zeta} & 0_{N,B} \cdot Y \\
 \nu \cdot X \Downarrow & & \Downarrow \nu \cdot Y \\
 (E \cdot N) \cdot X & \xrightarrow{\alpha} E \cdot (N \cdot X) \xrightarrow{E \cdot \varphi} E \cdot (N \cdot Y) \xrightarrow{\alpha^{-1}} & (E \cdot N) \cdot Y
 \end{array}$$

there exists a unique $\psi: X \Longrightarrow Y$ such that

$$N \cdot \psi = \varphi.$$

The fiber sequence: proof

