GOOD

ACTIONS

Algebraic and categorical perspectives

$$\begin{tikzcd} & B \arrow[d, shift left] \\ X \arrow[r] & E \arrow[u, shift left] \end{tikzcd}$$

01

Giuseppe Metere

Mathematical notes

Good Actions

Algebraic and categorical perspectives

These notes arose from an introductory lecture I gave at Stellenbosch University in April 2026, during a visit funded by NITheCS, whose support I gratefully acknowledge.

Giuseppe Metere

Map of the booklet

I

$$\begin{tikzcd}
 & B \arrow[d, shift left] \\
X \arrow[r] & E \arrow[u, shift left]
\end{tikzcd}$$

PART I

PART I

Actions in algebra

From transformation to split extension

GOOD ACTIONS

Giuseppe Metere

1

Actions in algebra

> *"Groups really shine when you let them act on something."*
>
> *P. Aluffi*, Algebra: Chapter 0.

A guiding idea for the first part of this note is that *external operations make structure visible*.

An algebraic object, left to itself, can look self-contained; once it is allowed to act on something else, its internal features suddenly become legible. This is a *leitmotif* in physics, and in science in general: understanding via interaction. In the classical scene a group acts on a set. From there the stage widens: the set may itself carry algebraic structure, becoming a group, a ring, or eventually a more exotic structure, such as a *hoop*.

As announced, we shall start with groups.

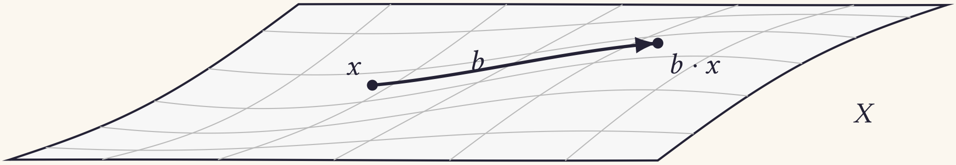
If B is a group and X is a set, a left action is a map

$$B \times X \longrightarrow X, \qquad (b, x) \longmapsto b \cdot x,$$

such that

$$1 \cdot x = x, \qquad b_1 \cdot (b_2 \cdot x) = (b_1 b_2) \cdot x.$$

The following picture represents dynamically the action of the element $b \in B$ which *sends* the element x to the element $b \cdot x$ of X:



To emphasise this viewpoint, one can speak of a B-*set*, so that one would describe action theoretical notions in geometrical terms. For instance, the orbits of the action correspond to the connected components of the B-set X.

Example 1.1. The definition of action given above is more precisely called *left action*. Of course one can define similarly *right actions*. For instance, in the definition of a n-dimensional affine space $\mathbb{A}$ over the reals, one can consider the right action of the vector space $\mathbb{R}^n$ on the set of points of $\mathbb{A}$:

$$\mathbb{A} \times \mathbb{R}^n \longrightarrow \mathbb{A}, \qquad (P, \mathbf{v}) \longmapsto P + \mathbf{v}.$$

Well... I am cheating a bit. This is in fact the action of a vector space, rather than that of a group, and has some special features I am not discussing here. However, the underlying abelian group $(\mathbb{R}^n, +, \mathbf{0})$ acts on the points of $\mathbb{A}$ in the sense specified above. Actually, the point here is that, endowed with this action, the set $\mathbb{A}$ is *shaped* by the action of $\mathbb{R}^n$: its shape is the affine structure.

Let us comment briefly on the axioms of action. Indeed, they seem to be quite natural: the first axiom says that the identity of the group acts trivially, while the second says that successive actions compose according to the multiplication of B. If we pretend x is an element of the group B, the axioms model the usual identity and the associativity properties of the group.

However, as educated mathematicians, we cannot be completely satisfied by this explanation. Analogies often support definitions, but they do not justify them conceptually. There is a second point of view, and it opens the door to category theory.

The action above is equivalently described as a homomorphism

$$\sigma: B \to \mathrm{Sym}(X), \qquad b \mapsto (x \mapsto b \cdot x),$$

and all actions arise this way. This makes group actions on sets an internal gadget in the category of groups, or in other words, makes it possible to express such action with the language of group theory. On the other hand, this still misses the point to make natural a generalisation to other algebraic structures.

To the rescue, here is a third point of view : actions are representations of B in the category of sets. More conceptually, if one regards B as the one-object category $B(*)$, then a B-set is just a functor

$$B(*) \longrightarrow \mathbf{Set}.$$

This simple observation already contains a remarkable part of the story: category theory enters as the natural language in which these various kinds of actions can be compared.

So the guiding question is the following: once the ambient category changes, what should replace the familiar action of a group on a set? For groups acting on groups the answer is classical and concrete. For commutative rings, unital rings, and Wajsberg hoops the answer becomes less evident, and eventually the correct notion emerges when one looks at split extensions.

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Group actions of groups on groups

Let B and X be groups. An action of B on the underlying set of X becomes an action *by group automorphisms* as soon as each translation

$$X \to X, \qquad x \mapsto b \cdot x,$$

preserves the group law of X. Writing the operation of X additively, this means

$$b \cdot (x_1 + x_2) = b \cdot x_1 + b \cdot x_2.$$

In particular,

$$b \cdot 0 = b \cdot (0 + 0) = b \cdot 0 + b \cdot 0,$$

so cancellation gives $b \cdot 0 = 0$. Hence each $b \in B$ acts by an endomorphism of X; since b^{-1} acts as its inverse, the action is in fact by automorphisms. Equivalently, a group action of B on the group X is the same thing as a homomorphism

$$B \longrightarrow \mathrm{Aut}(X).$$

Examples and morphisms

The archetypal example is conjugation. If $N \trianglelefteq B$ is a normal subgroup, then

$$B \times N \to N, \qquad (b, n) \longmapsto bnb^{-1},$$

is an action.

If $B \times X \to X$ and $C \times Y \to Y$ are actions, a morphism of actions is a pair of homomorphisms

$$g : B \to C, \qquad f : X \to Y,$$

such that

$$f(b \cdot x) = g(b) \cdot f(x) \qquad \text{for all } b \in B,\ x \in X.$$

Diagrammatically, this means that the following diagram of sets and functions commutes:

$$\begin{tikzcd}
B \times X \arrow[r] \arrow[d, "g\times f"'] & X \arrow[d, "f"] \\
C \times Y \arrow[r] & Y
\end{tikzcd}$$

A special situation is obtained by letting $g = 1_B$. In this case, $f : X \to Y$ is called B-equivariant homomorphism.

From actions to split epimorphisms

Given an action of B on X, one may package it into a single group by placing X and B side by side and defining a multiplication on $X \times B$ by

$$(x_1, b_1)(x_2, b_2) = \big(x_1 + b_1 \cdot x_2,\ b_1 b_2\big).$$

The identity element is $(0, 1)$, and the inverse of (x, b) is

$$(x, b)^{-1} = \big(b^{-1} \cdot (-x),\ b^{-1}\big).$$

A straightforward computation then shows associativity:

$$\begin{aligned}
\big((x_1, b_1)(x_2, b_2)\big)(x_3, b_3) &= \big(x_1 + b_1 \cdot x_2 + (b_1 b_2) \cdot x_3,\ b_1 b_2 b_3\big),\\
(x_1, b_1)\big((x_2, b_2)(x_3, b_3)\big) &= \big(x_1 + b_1 \cdot x_2 + b_1 \cdot (b_2 \cdot x_3),\ b_1 b_2 b_3\big),
\end{aligned}$$

and these coincide by the action axiom. The resulting group is the semidirect product $X \rtimes B$.

The projection and section are

$$\pi_2 : X \rtimes B \to B, \qquad (x, b) \mapsto b,$$

$$\iota_2 : B \to X \rtimes B, \qquad b \mapsto (0, b),$$

with $\pi_2 \iota_2 = 1_B$. Thus every action determines a split epimorphism.

From split epimorphisms to actions

Conversely, begin with a split epimorphism of groups

$$\begin{tikzcd} A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd} \qquad \text{with } ps = 1_B,$$

and let $X = \ker p$. Then X is a normal subgroup of A, and one defines

$$B \times X \to X, \qquad (b, x) \longmapsto s(b) + x - s(b),$$

that is, conjugation by the chosen section. Since X is normal, this element again lies in X. The action axioms follow immediately from $s(1) = 0$ and from the homomorphicity of s.

One thus recovers the correspondence:

$$\{B\text{-actions on groups } X\} \longleftrightarrow \{\text{split exact sequences over } B \text{ with kernel } X\}.$$

Moreover, this correspondence is functorial with respect to morphisms of actions.

Affine transformations

A particularly vivid example is furnished by affine transformations of $\mathbb{R}^n$. An affine map has the form

$$y = A\mathbf{x} + \mathbf{b}, \qquad A \in GL(n, \mathbb{R}), \ \mathbf{b} \in \mathbb{R}^n,$$

and may be written as the block matrix

$$\begin{bmatrix} A & \mathbf{b} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

acting on $\binom{x}{1}$. Matrix multiplication gives

$$\begin{bmatrix} A_1 & \mathbf{b}_1 \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} A_2 & \mathbf{b}_2 \\ \mathbf{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} A_1 A_2 & A_1 \mathbf{b}_2 + \mathbf{b}_1 \\ \mathbf{0}^\top & 1 \end{bmatrix},$$

which is precisely the semidirect-product law

$$(\mathbf{b}_1, A_1)(\mathbf{b}_2, A_2) = \big(\mathbf{b}_1 + A_1 \mathbf{b}_2,\ A_1 A_2\big).$$

Therefore

$$\mathrm{Aff}(n, \mathbb{R}) \cong \mathbb{R}^n \rtimes GL(n, \mathbb{R}).$$

Here $\mathbb{R}^n$ records translations, while $GL(n, \mathbb{R})$ records the linear part: scalings, rotations, and shears.

3

Actions in commutative rings

We now pass to commutative rings, at first without unit. Here the story changes register. If B and X are commutative rings in $\mathbf{CRng}$, the correct analogue of an external action is no longer a bare map $B \times X \to X$ on underlying sets, but a bilinear operation that respects the multiplicative texture of both rings.

Recall that the tensor product $B \otimes_{\mathbb{Z}} X$ is the commutative ring whose underlying abelian group is the tensor product of the additive groups of B and X, and whose multiplication is determined by

$$(b_1 \otimes x_1)(b_2 \otimes x_2) = b_1 b_2 \otimes x_1 x_2$$

and extended by bilinearity.

Definition 3.1. A *B-action on the commutative ring X* is a ring homomorphism

$$\xi : \ B \otimes X \longrightarrow X.$$

Writing $b \cdot x \ := \xi(b \otimes x)$, the axioms read

$$b_1 \cdot (b_2 \cdot x) = (b_1 b_2) \cdot x,$$

$$b \cdot (x_1 x_2) = (b \cdot x_1) x_2 = x_1 (b \cdot x_2).$$

Morphisms, again, are equivariant homomorphisms of rings.

The module case

If X has zero multiplication, that is, $x_1x_2 = 0$ for all $x_1, x_2 \in X$, then the last condition becomes vacuous and one recovers the usual notion of a B-module. Indeed, in that case a B-action satisfies

$$(b_1 + b_2) \cdot x = b_1 \cdot x + b_2 \cdot x, \qquad b \cdot (x_1 + x_2) = b \cdot x_1 + b \cdot x_2, \qquad b_1 \cdot (b_2 \cdot x) = (b_1 b_2) \cdot x.$$

Note that the first two equations express precisely the bilinearity of ξ.

Ordinary module theory therefore appears as a limiting case of ordinary actions.

From actions to split epimorphisms

Given a ring B-action $\xi : B \otimes X \to X$, one can endow a multiplication operation on the abelian group $X \oplus B$ by letting

$$(x_1, b_1)(x_2, b_2) = \big(x_1x_2 + b_1 \cdot x_2 + b_2 \cdot x_1,\ b_1 b_2\big).$$

Distributivity follows from bilinearity, and associativity follows from the axioms above. The ring obtained in this way is called the semidirect product of X and B with respect to the action ξ, and will again be denoted $X \rtimes B$.

The projection onto the second factor,

$$\pi_2 : X \oplus B \to B, \qquad (x, b) \mapsto b,$$

is a ring homomorphism, and the section

$$\iota_2 : B \to X \oplus B, \qquad b \mapsto (0, b)$$

is again a ring homomorphism.

From split epimorphisms to actions

Conversely, suppose that

$$\begin{tikzcd} A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

is a split epimorphism in $\mathbf{CRng}$, and let $X = \ker p$. Since kernels in commutative rings are ideals, X is an ideal of A. Define

$$b \cdot x := s(b)x.$$

Because X is an ideal, this belongs to X. Bilinearity is immediate:

$$(b_1 + b_2) \cdot x = s(b_1 + b_2)x = s(b_1)x + s(b_2)x = b_1 \cdot x + b_2 \cdot x,$$
$$b \cdot (x_1 + x_2) = s(b)(x_1 + x_2) = b \cdot x_1 + b \cdot x_2.$$

Moreover,

$$b_1 \cdot (b_2 \cdot x) = s(b_1)s(b_2)x = s(b_1 b_2)x = (b_1 b_2) \cdot x,$$

and multiplicativity gives

$$(b_1 \cdot x_1)(b_2 \cdot x_2) = s(b_1)x_1 s(b_2)x_2 = s(b_1 b_2)x_1 x_2 = (b_1 b_2) \cdot (x_1 x_2).$$

Hence $B \otimes X \to X$ is a ring homomorphism.

As in the group case, one obtains a bijective correspondence between actions of B on X and split epimorphisms over B with kernel X.

4

Actions of unital rings

The unital case is especially illuminating, because it compels us to compare two categories:

$$\mathbf{Rng} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Ring},$$

where U is the forgetful functor and F adjoins a multiplicative unit.

Suppose

$$\begin{tikzcd} A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

is a split epimorphism in $\mathbf{Ring}$, and let $X = \ker p$. Then X is usually not a unital ring, although it is still a ring and still carries an action of B. Indeed, one can check

$$1 \cdot x = s(1)x = 1_A x = x.$$

This motivates the following definition.

Definition 4.1. If B is a unital commutative ring and X is a commutative ring, a *unital action* of B on X is a B-action such that

$$1 \cdot x = x \qquad \text{for all } x \in X.$$

Thus a unital ring may perfectly well act on a ring with no unit of its own, but the action still remembers the unit of the acting ring.

Adjoining a unit and the Dorroh extension

The left adjoint $F : \mathbf{Rng} \to \mathbf{Ring}$ admits a concrete description:

$$F(X) = X \times \mathbb{Z},$$

with multiplication

$$(x_1, m_1)(x_2, m_2) = \big(x_1x_2 + m_1x_2 + m_2x_1,\ m_1m_2\big).$$

The element $(0, 1)$ is the multiplicative unit. Indeed,

$$(x, m)(0, 1) = \big(0 + 0 + x,\ m\big) = (x, m),$$

and similarly on the other side. The subset $X \times \{0\}$ is an ideal, naturally identified with X, and it is the kernel of the projection

$$F(X) = X \rtimes \mathbb{Z} \xrightarrow{\pi_2} \mathbb{Z}.$$

This is the classical *Dorroh extension*, named after J.Ł. Dorroh[1].

One obtains the canonical action of $\mathbb{Z}$ on X by repeated addition:

$$(m, x) \longmapsto x + x + \cdots + x = mx.$$

Taking the associated semidirect product recovers exactly the ring $X \rtimes \mathbb{Z}$ above.

Categorically, this says something striking: every non-unital ring can be exhibited as the kernel of a split epimorphism onto $\mathbb{Z}$, and one has an equivalence of categories

$$(\mathbf{Ring} \downarrow \mathbb{Z}) \simeq \mathbf{Rng}.$$

Since $\mathbb{Z}$ is the initial object of $\mathbf{Ring}$, this gives the ring-theoretic template that will later reappear for Wajsberg hoops and Wajsberg algebras.

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[1] J.L. Dorroh: Concerning adjunctions to algebras. Bull. Amer. Math. Soc. (38) 1932.

5

Wajsberg hoops and Wajsberg algebras

We now cross over to the last character of our story. The atmosphere changes, but the conceptual frame remains recognizably the same.

Let us introduce a remarkable variety of hoops, namely that of Wajsberg Hoops, a variety of algebras relevant to substructural logics[2]. A convenient presentation of Wajsberg hoops is the following.

> **Definition 5.1.** A *Wajsberg hoop* is an algebra $(X, \cdot, \to, 1)$ such that $(X, \cdot, 1)$ is a commutative monoid and, for all $x, y, z \in X$,
>
> (H1) $x \to x = 1$;
>
> (H2) $(x \to y) \cdot x = (y \to x) \cdot y$;
>
> (H3) $xy \to z = x \to (y \to z)$;
>
> (H4) $(x \to y) \to y = (y \to x) \to x$.

Axiom (H3) is the residuation law in equational form, while (H4) is the distinctive Wajsberg symmetry. Two useful derived identities are

$$x \to 1 = 1, \qquad 1 \to x = x.$$

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[2] Substructural logics are logics in which one or more of the usual structural rules for assumptions are absent, restricted, or changed.

The natural order

Define

$$x \le y \quad\Longleftrightarrow\quad x \to y = 1.$$

Then $\le$ is a partial order. Reflexivity is exactly (H1). If $x \le y$ and $y \le x$, then by (H2)

$$x = (x \to y) \cdot x = (y \to x) \cdot y = y,$$

so antisymmetry holds. For transitivity, assume $x \le y$ and $y \le z$. Then

$$x \to z = 1 \cdot (x \to z) = ((x \to y) \cdot x) \to z.$$

Using (H2) and then (H3), one gets

$$((x \to y) \cdot x) \to z = ((y \to x) \cdot y) \to z = (y \to x) \to (y \to z) = (y \to x) \to 1 = 1,$$

so $x \le z$. The element 1 is the top element, because $x \to 1 = 1$ for every x. In general there need not be a bottom element.

Lattice operations and residuation

The meet is given by

$$x \wedge y \ := (x \to y) \cdot x = (y \to x) \cdot y,$$

where the equality is (H2). The join may be written as

$$x \vee y \ := (x \to y) \to y.$$

Thus every Wajsberg hoop is in fact a lattice with top 1.

Moreover, the multiplication is residuated: by (H3),

$$xy \le z \quad\Longleftrightarrow\quad xy \to z = 1 \quad\Longleftrightarrow\quad x \to (y \to z) = 1 \quad\Longleftrightarrow\quad x \le y \to z.$$

So Wajsberg hoops are precisely prelinear commutative integral residuated monoids in which the Wajsberg identity holds. In particular, prelinearity means

$$(x \to y) \vee (y \to x) = 1.$$

Bounded versus cancellative hoops

A recurrent motif is that Wajsberg hoops often gravitate toward two opposite poles: either they are bounded, or they are cancellative. Bounded examples are precisely the Wajsberg algebras introduced below. Cancellative examples typically arise from negative cones of linearly ordered abelian groups.

> **Definition 5.2.** A *Wajsberg algebra* is an algebra $(B, \cdot, \to, 1, 0)$ such that $(B, \cdot, \to, 1)$ is a Wajsberg hoop and
>
> $$0 \to x = 1 \qquad \text{for all } x \in B.$$
>
> Equivalently, $0 \leq x$ for every x, so a Wajsberg algebra is a bounded Wajsberg hoop.

Examples

Typical examples are:

(1) the real interval $[0, 1]$ with Łukasiewicz operations

$$x \cdot y = \max\{0, x + y - 1\}, \qquad x \to y = \min\{1, 1 - x + y\};$$

(2) the finite chains $L_n = \Gamma(\mathbb{Z}, n) = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$;

(3) the rational interval $[0, 1] \cap \mathbb{Q}$;

(4) intervals $\Gamma(G, u)$ in lattice-ordered abelian groups with strong unit u;

(5) negative cones of linearly ordered abelian groups, such as $\mathbb{Z}_{\leq 0}$ or $\mathbb{R}_{\leq 0}$, which provide typical unbounded, hence cancellative, examples.

6

What about hoop actions?

After the ring case, the obvious question is whether a comparable story survives for Wajsberg hoops. It does, and in a remarkably suggestive form, although a full explicit description of arbitrary actions is not available.

There is a forgetful functor

$$U: \mathbf{WAlg} \longrightarrow \mathbf{WHoops}$$

which simply forgets the bottom element. Its left adjoint adjoins a bottom element, and the initial object of $\mathbf{WAlg}$ is the two-element algebra

$$\mathbf{2} = \{0, 1\},$$

whereas the one-element hoop $\{1\}$ is initial in $\mathbf{WHoops}$.

For a Wajsberg hoop X, the free bounded object is

$$F(X) = X \rtimes \mathbf{2}.$$

The formulas are easiest to read by separating the two components. If $x_1, x_2 \in X$, then the operations on $X \rtimes \mathbf{2}$ are given by

$$\begin{aligned}
(x_1, 1) \to (x_2, 1) &= (x_1 \to x_2, 1),\\
(x_1, 1) \to (x_2, 0) &= (x_1 \cdot x_2, 0),\\
(x_1, 0) \to (x_2, 1) &= (x_1 \oplus x_2, 1),\\
(x_1, 0) \to (x_2, 0) &= (x_2 \to x_1, 1),
\end{aligned}$$

where

$$x_1 \oplus x_2 \ := (x_1 \to x_1x_2) \to x_2.$$

The multiplication is

$$\begin{aligned}(x_1, 1) \cdot (x_2, 1) &= (x_1x_2, 1),\\ (x_1, 1) \cdot (x_2, 0) &= (x_1 \to x_2, 0),\\ (x_2, 0) \cdot (x_1, 1) &= (x_1 \to x_2, 0),\\ (x_1, 0) \cdot (x_2, 0) &= (x_1 \oplus x_2, 0).\end{aligned}$$

Finally,

$$0_{F(X)} = (1, 0), \qquad 1_{F(X)} = (1, 1).$$

These formulas encode the same phenomenon already encountered in the Dorroh extension: the adjunction freely inserts the missing distinguished constant and places X inside a bounded object as the kernel $X \times \{1\}$ of the projection onto $\mathbf{2}$.

In particular:

(1) $F(X) = X \times \mathbf{2}$ is a bounded Wajsberg hoop, hence a Wajsberg algebra;

(2) the unit $\eta_X : X \to UF(X)$ is given by $x \mapsto (x, 1)$, so that X is canonically identified with the ideal $X \times \{1\}$;

(3) one has an equivalence

$$(\mathbf{WAlg} \downarrow \mathbf{2}) \simeq \mathbf{WHoops}.$$

This is the precise analogue of the ring equivalence

$$(\mathbf{Ring} \downarrow \mathbb{Z}) \simeq \mathbf{Rng}.$$

In both cases one starts from a category with a distinguished constant, forgets that constant, and recovers the original objects as ideals or kernels of split extensions over the initial algebra of the richer category. The parallel may be summarized as follows:

$$\mathbf{Rng} \simeq (\mathbf{Ring} \downarrow \mathbb{Z}), \qquad \mathbf{WHoops} \simeq (\mathbf{WAlg} \downarrow \mathbf{2}).$$

This parallel is one of the motivations for the second, more categorical part of the seminar: it suggests that actions should be studied in terms of split epimorphisms over the initial object, with the preservation of 1 in rings playing the same role as the preservation of 0 in Wajsberg algebras.

II

$$\begin{tikzcd}
 & B \arrow[d, shift left] \\
X \arrow[r] & E \arrow[u, shift left]
\end{tikzcd}$$

PART II

PART II

Intermezzo: adjoining 0 versus adjoining 1

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A pleasant way into the categorical part is by means of a short historical and conceptual détour.

In the classical ring-theoretic world, one often begins with a ring without unit and then adjoins 1. In the world of Wajsberg hoops, by contrast, the decisive move goes the other way: one starts from a structure with top element 1, but with no bottom element, and then adjoins 0. Formally these are operations in quite different algebraic universes, yet they play curiously parallel rôles. In both cases one freely adds a distinguished constant, and by doing so one alters the surrounding landscape in a decisive way.

This parallel is not only algebraically suggestive; it also has a certain intellectual charm. The constants 0 and 1 come laden with mathematical history, and they do not enter that history symmetrically. Zero arrived comparatively late, through Indian mathematics[3] and then through the Arabic world into Latin Europe[4], whereas the unit was conceptually present earlier. Indeed, philosophical disputes on the nature of numbers, have often focused on zero, whose *existence* was somehow more difficult to accept, than that of the other natural numbers. There is a Latin locution for this: *horror vacui*, meaning *fear of emptiness*, which traces back to Aristotle. As a matter of fact, Giuseppe Peano, Italian mathematician who introduced the axioms of natural numbers[5] begins the natural numbers with 1. However, not even 1 has been always considered as a number. It was not regarded as such by many Greek philosophers, such as Aristotle himself. Indeed, 1 was not considered as a proper number: numbers were rather defined as a quantity of ones, or of *monads*, so that 2 was the first actual number! This conception lasted for quite a long time. In the late Roman polymath Boethius (480–524 CE), based on the Neopythagorean tradition,

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[3] Zero was introduced in Brahmasphuta Siddhānta, by the Indian mathematician Brahmagupta, 628 CE.

[4] The Islamic mathematician Muhammad ibn Musa al-Khwarizmi was active in Baghdad around 820 CE. Latin translations of al-Khwarizmi's textbook on Indian arithmetic (Algorithmo de Numero Indorum) were crucial 12th-century works that introduced the decimal positional number system and Hindu-Arabic numerals to Europe. In 1202, and then later in 1228, Leonardo Bonaccio, known as Fibonacci, published the *Liber abbaci*, a work in fifteen chapters in which he introduced the nine digits he called *Indian*, together with the symbol 0.

[5] Giuseppe Peano, Arithmetices principia, nova methodo exposita, Torino, 1889.

still upheld this idea[6].

The historical details matter less than the impression they leave behind: neither 0 nor 1 is merely an ordinary constant. Each marks a threshold in mathematical thought. This observation becomes a guiding metaphor for what follows. To adjoin 1 to a non-unital ring is to move from a world where multiplication has no global identity to one where such an identity exists and governs the structure. To adjoin 0 to a Wajsberg hoop is to move from a setting with a top element but no guaranteed bottom to a bounded setting in which the order-theoretic picture becomes more complete. In each case, the passage is not merely cosmetic: it produces a new ambient category and opens the way to describing the original structures as kernels, ideals, or fibres of split extensions.

This is the thread that ties together the seemingly different topics of the talk. The first half develops the familiar story of actions and split extensions for groups and rings. The second half develops an analogous picture in the setting of Wajsberg hoops and Wajsberg algebras. The historical contrast between adjoining 1 and adjoining 0 provides an elegant way to anticipate the punchline: the two theories are not identical, but they mirror one another closely enough to suggest a common categorical pattern.

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[6] Anicius Manlius Severinus Boethius, De arithmetica, ca. 500 CE.

III

$$\begin{tikzcd}
 & B \arrow[d, shift left] \\
X \arrow[r] & E \arrow[u, shift left]
\end{tikzcd}$$

PART III

PART III

Good categorical actions

Points, kernels, ideals and semidirect products
Coherence, ideality and BAT

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7

Categorical preliminaries: points, change of base, and semidirect products

From this point on, the exposition becomes more categorical. The aim is not to develop the full theory, but to isolate the conceptual mechanism behind the examples above.

We now step back and view the whole discussion from a wider categorical balcony before returning, with better light, to the theory of actions.

We start with definitions and notions which are now classics in contemporary categorical algebra. The interested reader may consult [2] for a comprehensive account of the matter.

Definition 7.1. Let $\mathbb{C}$ be a category with finite limits and let $B \in \mathbb{C}$. The category of *points over* B, denoted by $\mathrm{Pt}_{\mathbb{C}}(B)$, has as objects the split epimorphisms

$$\begin{tikzcd} A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd} \qquad \text{with } ps = 1_B.$$

A morphism $f : (A, p, s) \to (A', p', s')$ of points over B is given by a morphism $f : A \to A'$ in $\mathbb{C}$ such that $p'f = p$ and $s' = fs$.

A simple but crucial point is that, in a sense, points are the most economical internal avatars of actions. In groups and rings, actions are encoded by semidirect products, and semidirect products are nothing but split epimorphisms wearing algebraic clothes.

Now let $f : E \to B$ be a morphism in $\mathbb{C}$. If pullbacks along f exist, then one has the familiar change-of-base functor

$$f^* : \mathrm{Pt}_{\mathbb{C}}(B) \longrightarrow \mathrm{Pt}_{\mathbb{C}}(E),$$

obtained by pulling back a split epimorphism over B along f.

If the change-of-base functor f^* admits a left adjoint, we denote it by

$$f_! : \mathrm{Pt}_{\mathbb{C}}(E) \longrightarrow \mathrm{Pt}_{\mathbb{C}}(B).$$

Definition 7.2 ([5]). A finitely complete category $\mathbb{C}$ is *protomodular* if for every morphism f the change-of-base functor f^* is conservative.

Definition 7.3 ([8]). A category is *semi-abelian* if it is pointed, protomodular, Barr-exact, and has finite coproducts.

One may also describe semidirect products in this language.

Definition 7.4 ([6], see also [3]). Assume that for every f the pullback functor f^* on points has a left adjoint $f_!$. Then $\mathbb{C}$ is said to *have semidirect products* if each f^* is monadic.

Let us unfold this definition. Say T^f is the monad associated with the right adjoint f^*. The functor f^* being monadic means that the canonical comparison $K : \mathrm{Pt}_{\mathbb{C}}(B) \to \mathrm{Pt}_{\mathbb{C}}(E)^{T^f}$ is an equivalence of categories.

$$\begin{tikzcd}
 & (X,\xi) \in \arrow[dl, dashed, mapsto] & \mathrm{Pt}_{\mathbb{C}}(E)^{T^f} \arrow[d, "\text{forgetful}"] \\
(X,\xi) \rtimes (B,f) \in & \mathrm{Pt}_{\mathbb{C}}(B) \arrow[ur, "K"] \arrow[r, "f^*"'] \arrow[r, phantom, bend left=15, "\perp"] & \mathrm{Pt}_{\mathbb{C}}(E) \arrow[l, bend right=30, "f_!"']
\end{tikzcd}$$

The semidirect product of the T^f-algebra (X, ξ) with B along f is then an essential preimage of (X, ξ) along K. This is represented in the picture above, where essential preimage is depicted as a dashed assignment.

The problem with this framework is that it only allows us to express points over B as algebras on points over E, which is not a big deal, unless the latter are somehow easier to deal with than the former. This viewpoint is taken in the following section.

8

From pointed categories to kernels

When the ambient category is pointed, the theory of points over B can be compressed further and recast in terms of kernels. Let $0 \to B$ be the zero morphism. Then the fibre over the zero object already contains the essential information, since $\mathrm{Pt}_{\mathbb{C}}(0) \simeq \mathbb{C}$.

More concretely, in a pointed category with enough structure one may compare

$$\mathrm{Pt}_{\mathbb{C}}(B) \qquad \text{and} \qquad \mathrm{Pt}_{\mathbb{C}}(0)^{B\flat},$$

where the latter notation stands for the category of algebras for the monad induced by the base change along $0 \to B$. In the presence of a zero object, one may summarize this by saying that split epimorphisms over B are equivalent to *algebras over their kernels*. This is depicted in the following diagram:

$$\begin{tikzcd}
 & (X,\xi) \;\in \arrow[dl, dashed, mapsto] & & \mathbb{C}^{B\flat} \arrow[d, "\text{forgetful}"] \\
(X,\xi)\rtimes B \;\in & & \mathrm{Pt}_{\mathbb{C}}(B) \arrow[ur, "K"] \arrow[r, "\text{ker}"'] & \mathbb{C} \arrow[l, bend right, "B+"'] \arrow[l, phantom, bend right=15, "\perp"']
\end{tikzcd}$$

Notice that, since we are making a canonical choice for the change-of-base, namely the initial arrow ι_B, we dropped f from notation, and let $B\flat := T^{\iota_B}$.

Algebras for $B\flat$ are, as usual, morphisms

$$\xi : B\flat X \to X$$

satisfying axioms that are expressed by the commutativity of suitable diagrams. Now the point here is that these are in fact *morphisms* in the base category $\mathbb{C}$. Thus this notion allows us to express in the internal language of the category the notion of action in many categories of interest, and their description is functorial. This addresses one of the issues we exposed in the first part.

9

Relative ideals and augmentation ideals

The next step is to translate this categorical insight into an adjoint situation,

$$\begin{tikzcd}
\mathbb{U} \arrow[r, "U"'] \arrow[r, phantom, shift left=3, "\perp"] & \mathbb{V} \arrow[l, bend right=40, "F"']
\end{tikzcd}$$

where $\mathbb{V}$ is semi-abelian, U is conservative, and in the examples U forgets some extra structure. The guiding question is this: how does one recognize, internally, those subobjects of UA that genuinely arise as kernels of morphisms coming from $\mathbb{U}$?

> **Definition 9.1** ([10]). A monomorphism $x : X \to UA$ is a *relative* U*-ideal* if there exists a morphism $p : A \to B$ in $\mathbb{U}$ such that x is the kernel of Up in $\mathbb{V}$.

Let 0 denote the zero object of $\mathbb{V}$, and put

$$I := F(0),$$

so that I is the initial object of $\mathbb{U}$. For every $X \in \mathbb{V}$, let

$$p_X := F(!_X) : F(X) \longrightarrow I,$$

where $!_X : X \to 0$ is the unique morphism.

Definition 9.2 ([10]). The unit component $\eta_X : X \to UF(X)$ is an *augmentation U-ideal* if it is the kernel of $U(p_X)$:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "0"'] & UF(X) \arrow[d, "U(p_X)"] \\
& U(I).
\end{tikzcd}$$

Notice that, when η_X is an augmentation U-ideal for every X, pullback cancellation implies that η is a cartesian natural transformation.

Then the following fact can be proved.

Theorem 9.3 ([10], [9]). *Assume that every unit component $\eta_X : X \to UF(X)$ is an augmentation U-ideal. Then the kernel functor*

$$\mathrm{Ker} : (\mathbb{U} \downarrow I) \longrightarrow \mathbb{V}$$

is an equivalence of categories.

10

Ideally exact categories and ideally exact contexts

The next step is to return to categories and isolate the precise environment in which the later action theory is meant to live.

> **Definition 10.1.** Following [9], a category $\mathbb{U}$ is *ideally exact* if it is Barr-exact, protomodular, has finite coproducts, and the unique morphism from the initial object to the terminal object is a regular epimorphism.

The notion of ideally exact category determines a convenient setting to deal with the different kinds of actions discussed in the previous sections. This is not immediately evident from the definition above, but [9] gives several characterisations, from which one can state the following fact: if a category $\mathbb{U}$ is ideally exact, then there exists a monadic functor

$$U : \mathbb{U} \to \mathbb{V}$$

with $\mathbb{V}$ semi-abelian such that the unit $\eta : 1_{\mathbb{V}} \to UF$ is cartesian.

This is the point at which the later terminology begins to earn its keep.

> **Definition 10.2.** An *ideally exact context* is an adjunction
>
> $$F \dashv U : \mathbb{U} \to \mathbb{V}$$
>
> where $\mathbb{U}$ is ideally exact, $\mathbb{V}$ is semi-abelian, U is monadic, and the unit $\eta : 1_{\mathbb{V}} \to UF$ is cartesian.

From this point onward one works systematically inside such a context.

11

Coherent actions and ideal actions

Let us now fix an ideally exact context

$$F \dashv U : \mathbb{U} \to \mathbb{V}$$

and an object $B \in \mathbb{U}$. The general problem is now to compare points over B in $\mathbb{U}$ with suitable action data over $U(B)$ in $\mathbb{V}$.

Two notions now enter the picture: coherent actions and ideal actions. The first one is defined directly in $\mathbb{V}$, while the second remembers whether the action actually comes from a split epimorphism in $\mathbb{U}$.

To formulate this, let 0 denote the zero object of $\mathbb{V}$. For $X \in \mathbb{V}$, write

$$\tau_X : X \longrightarrow 0, \qquad \iota_X : 0 \longrightarrow X$$

for the unique morphisms, and let

$$\iota_B : F(0) \longrightarrow B$$

be the unique morphism from the initial object of $\mathbb{U}$. There is a canonical action

$$\xi_0 : UF(0) \flat X \longrightarrow X,$$

associated with the split epimorphism

$$\begin{tikzcd}
UF(X) \arrow[r, shift left, "UF(\tau_X)"] & UF(0). \arrow[l, shift left, "UF(\iota_X)"]
\end{tikzcd}$$

Definition 11.1 (Coherent actions). Let $\xi : U(B)\flat X \to X$ be an action in $\mathbb{V}$. We say that the action ξ is *coherent* if the following diagram commutes:

$$\begin{tikzcd}
U(B)\flat X \arrow[r, "\xi"] & X \\
UF(0)\flat X \arrow[u, "U(\iota_B)\flat \mathrm{id}_X"] \arrow[ur, "\xi_0"'] &
\end{tikzcd}$$

In algebraic terms, an action is coherent if it behaves coherently with the canonical action of $UF(0)$. Notice that $F(0)$ is the initial object in $\mathbb{U}$, so that when $\mathbb{U}$ is a variety of universal algebras, coherence refers precisely to the action of the constants.

Now, it is easy to translate the notion in terms of split epimorphism.

Lemma 11.2. *Given an action $\xi : U(B)\flat X \to X$, let*

$$\begin{tikzcd}
A \arrow[r, shift left, "p"] & U(B) \arrow[l, shift left, "s"]
\end{tikzcd}$$

be a split epimorphism associated with ξ. Then, ξ is coherent if and only if there exists a morphism $f : UF(X) \to A$ in $\mathbb{V}$ such that the following diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, equal] & UF(X) \arrow[r, shift left, "UF(\tau_X)"] \arrow[d, dashed, "f"'] & UF(0) \arrow[l, shift left, "UF(\iota_X)"] \arrow[d, "U(\iota_B)"] \\
X \arrow[r, "k"'] & A \arrow[r, shift left, "p"] & U(B) \arrow[l, shift left, "s"]
\end{tikzcd}$$

is a morphism of split extensions in $\mathbb{V}$. Furthermore, the square on the right is a split pullback.

This allows us to make the notion of coherence intrinsic to split epimorphisms.

Definition 11.3. A split epimorphism (A, p, s) is termed *coherent* if it satisfies the condition stated in the lemma above.

Although the definition is couched categorically, its meaning is rather concrete. In rings, coherence says that the image of the unit under the chosen section really behaves like a unit, exactly as in the Dorroh extension $F(X) = X \rtimes \mathbb{Z}$. In Wajsberg hoops, coherence expresses the parallel requirement for the adjoined bottom element in $F(X) = X \rtimes \mathbf{2}$.

Definition 11.4. A split epimorphism

$$\begin{tikzcd} A \arrow[r, shift left, "p"] & U(B) \arrow[l, shift left, "s"] \end{tikzcd}$$

in $\mathbb{V}$ is *ideal* if there exists a split epimorphism

$$\begin{tikzcd} A' \arrow[r, shift left, "p'"] & B \arrow[l, shift left, "s'"] \end{tikzcd}$$

in $\mathbb{U}$ and an isomorphism $\sigma : U(A') \to A$ carrying $U(p', s')$ to (p, s).
An action is *ideal* if it determines an ideal split epimorphism.

Definition 11.5. Let

$$h : (A_1, p_1, s_1) \longrightarrow (A_2, p_2, s_2)$$

be a morphism between ideal split epimorphisms over $U(B)$. It is an *ideal morphism* if, for suitable corresponding split epimorphisms

$$(A_i', p_i', s_i') \in \mathrm{Pt}_{\mathbb{U}}(B) \qquad (i = 1, 2)$$

and isomorphisms $\sigma_i : U(A_i') \to A_i$, there exists a morphism

$$h' : (A_1', p_1', s_1') \longrightarrow (A_2', p_2', s_2')$$

in $\mathrm{Pt}_{\mathbb{U}}(B)$ such that

$$h\,\sigma_1 = \sigma_2\,U(h').$$

An ideal morphism of ideal actions is a morphism corresponding to an ideal morphism of their associated split epimorphisms.

The following result establishes a connection between the two notions. Its proof relies essentially on protomodularity.

Theorem 11.6 ([11]). *In every ideally exact context, every ideal action is coherent.*

Remark 11.7. The converse does not hold in every ideally exact context. The contexts in which it does hold, together with the corresponding condition on morphisms, are singled out below.

12

BAT: Buona Action Theory

At this point a compact acronym becomes convenient.

Definition 12.1. An ideally exact context has a *good theory of actions*, abbreviated BAT[a], if every coherent action is ideal and every morphism between coherent actions is ideal.

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[a] The acronym BAT is inspired by the notion of BIT-variety, where BIT stands for **B**uona (good, in Italian) **I**deal **T**heory, introduced by A. Ursini. Analogously, BAT stands for **B**uona **A**ction **T**heory.

This is a useful notion because coherence is often far easier to verify directly: it can be expressed by cartesian or pullback conditions. The slogan is therefore attractive and practical at once: in a BAT context, ideality can be detected by checking coherence.

In the known examples

$$\mathbf{Rng} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Ring} \qquad \text{and} \qquad \mathbf{WHoops} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{WAlg},$$

coherent and ideal actions, as well as their morphisms, coincide [11].

13

Concluding picture

We began with familiar external actions and we are concluding in a purely categorical setting.

Actions make algebraic structure visible. Instead of studying an object only through its internal operations and identities, one may observe how it transforms another object. In this way, groups, rings, or algebras reveal their structure through their representations, modules, or more general actions.

Semidirect products turn this external viewpoint into an internal one. An action of B on X gives rise to a split epimorphism

$$X \rtimes B \longrightarrow B,$$

whose kernel is X and whose section remembers B. Thus actions and split epimorphisms provide two complementary languages for the same phenomenon. We introduced two key notions:

- that of *coherent action*, which is an action satisfying the necessary pullback compatibility with the free augmentation;
- that of *ideal action*, which is an action that really comes from a split epimorphism in the ideally exact category.

In algebraic contexts with distinguished constants, however, one wants to control how the free constants behave. This is the role of coherence. It ensures that the constant carried by the acting object interacts correctly with the object being acted upon, as happens for the unit in rings or the bottom element in related algebraic settings.

In every ideally exact context, every ideal action is coherent. A BAT context is one in which the converse also holds and morphisms between coherent actions are ideal. Hence, in a BAT context, coherent and ideal actions coincide, together with their morphisms.

References

[1] M. Abad, D. N. Castaño and J. P. Díaz Varela, *MV-closures of Wajsberg hoops and applications*, Algebra Universalis **64** (2010), 213–230.

[2] F. Borceux and D. Bourn, *Mal'cev, Protomodular, Homological and Semi-Abelian Categories*, Kluwer Academic Publishers, 2004.

[3] F. Borceux, G. Janelidze and G. M. Kelly, *Internal object actions*, Commentationes Mathematicae Universitatis Carolinae **46** (2005), no. 2, 235–255.

[4] B. Bosbach, *Komplementare Halbgruppen. Axiomatik und Arithmetik*, Fundamenta Mathematicae **64** (1969), 257–287.

[5] D. Bourn, *Normalization equivalence, kernel equivalence and affine categories*, in *Category Theory, Proceedings Como 1990*, A. Carboni, M. C. Pedicchio and G. Rosolini, eds., Lecture Notes in Mathematics, Vol. **1488**, Springer, 1991, 43–62.

[6] D. Bourn and G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory and Applications of Categories **4** (1998), no. 2, 37–46.

[7] D. Bourn and G. Janelidze, *Characterisation of protomodular varieties of universal algebras*, Theory and Applications of Categories **11** (2003), no. 6, 143–147.

[8] G. Janelidze, L. Márki and W. Tholen, *Semi-abelian categories*, Journal of Pure and Applied Algebra **168** (2002), no. 2, 367–386.

[9] G. Janelidze, *Ideally exact categories*, Theory and Applications of Categories **41** (2024), no. 11, 414-425.

[10] S. Lapenta, G. Metere and L. Spada, *Relative ideals in homological categories, with an application to MV-algebras*, Theory and Applications of Categories **41** (2024), no. 27, 878–893.

[11] M. Mancini, G. Metere and F. Piazza, *Coherent and ideal actions in ideally exact categories*, Theory and Applications of Categories **45** (2026), no. 31, 1280–1320.