A Dialogue on Eight Questions in One

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July 5, 2026

Abstract

A question raised by Paolo Aglianò asks whether there exist protomodular varieties, or prevarieties, in which the free algebra on the empty set does not exist. We discuss the question in the form of a short dialogue. The issue is not a single obstruction: the question suggests the separation of several nearby formulations. The answer depends on whether empty algebras are admitted, on whether one works with varieties or prevarieties, and on whether finite limits are assumed. If empty algebras are admitted, the non-trivial case is ruled out by a general argument with change of base along the empty algebra. In the non-empty reading, non-empty heaps give a protomodular variety, and hence a prevariety, with no initial object. The dialogue makes explicit how these answers coexist.

The dialogue

Cleon found Damon with Theon and Nicias. They were discussing a familiar disagreement: whether an algebra, when the signature permits it, may have the empty set as its universe.

Cleon interrupted them by recalling a question that Aglianò had raised in a survey on ideals in universal algebra[1].

Cleon. I have found a question which seems innocent enough:

> "There is a problem which, to the best of my knowledge, is still open: are there protomodular varieties with no initial objects? Which means are there protomodular (pre)varieties in which the free algebra over the empty set does not exist?"

What should the answer be?

Damon. Before answering, let us make sure that the question has only one meaning. What do you call an algebra?

Nicias. In the usual algebraic way: a non-empty set equipped with finitary operations of a fixed type.

Theon. That is one common convention. But there is another. If the signature has no nullary operations, the empty set carries the required operations uniquely; some authors therefore allow the empty algebra.

Damon. Then Cleon's question has already split in two. Under one convention the empty algebra is present when the signature permits it. Under the other it is not an algebra at all. We must not answer before knowing which category is being asked about.

[1] See [1, Remark 3.2.5].

CLEON. But what is, algebraically, the free algebra on the empty set?

NICIAS. In a variety, the usual candidate is the algebra $T_{\mathcal{V}}(\varnothing)$ of closed terms, that is, terms with no variables, modulo the identities of the variety.

THEON. When empty algebras are allowed, this is always the free algebra on the empty set. In the non-empty reading it is free precisely when it is non-empty. Categorically, when it exists, it is just the initial algebra: there is exactly one homomorphism from it to every algebra in the variety.

DAMON. So, if empty algebras are allowed and the signature has no constants, the empty algebra may itself be initial. If algebras are required to be non-empty, the same construction may produce the empty algebra, which is not then an admissible object.

CLEON. Then how does protomodularity enter?

THEON. By Bourn's definition[2], a category with split pullbacks is protomodular when, for every morphism $u\colon X \to B$, the change-of-base functor

$$u^*\colon \mathrm{Pt}_B(\mathcal{C}) \longrightarrow \mathrm{Pt}_X(\mathcal{C})$$

is conservative. Here $\mathrm{Pt}_B(\mathcal{C})$ is the category of split epimorphisms over B. Notice that finite limits are not required by this definition.

DAMON. Quite right, Theon. Still, finite limits are often assumed when one speaks of protomodular categories. We shall therefore keep the two issues separate, and look at the cases with and without finite completeness. Now let us return to the question.

THEON. For varieties, there is a natural first test: the Bourn–Janelidze term characterisation of protomodular varieties.

CLEON. What does that characterisation say?

THEON. It says that a variety $\mathcal{V}$ is protomodular if and only if, for some $n \geq 0$, there are nullary terms

$$e_1, \ldots, e_n,$$

binary terms

$$t_1, \ldots, t_n,$$

and an $(n+1)$-ary term t such that

$$t(x, t_1(x,y), \ldots, t_n(x,y)) = y$$

and

$$t_i(x,x) = e_i \qquad (1 \leq i \leq n).$$

This is the theorem of Bourn and Janelidze[3].

DAMON. So, in a protomodular variety, the theory itself tends to produce distinguished constants.

THEON. Exactly. If $n > 0$, those constants cannot be interpreted in the empty algebra. Thus, under the convention allowing empty algebras, a non-trivial protomodular variety cannot have the empty algebra as its initial object, unless the characterisation falls into the case $n = 0$.

[2] See [3].
[3] See [5].

CLEON. Still, I do not think we need the Bourn–Janelidze characterisation to settle the empty-initial case. There is a more general argument.

DAMON. General in what sense?

CLEON. It applies to prevarieties as well. Let a prevariety mean a class of algebras closed under isomorphisms, subalgebras and arbitrary products, that is, an ISP-class. As before, the word "subalgebra" is read according to the convention in force.

NICIAS. So first suppose that empty algebras are admitted.

CLEON. Exactly. If the signature admits the empty algebra and a prevariety contains a non-empty algebra, then the empty algebra is a subalgebra of it. Hence the empty algebra belongs to the prevariety and is initial.

THEON. That gives an initial object. But why should protomodularity force a collapse?

CLEON. Let 0 be the empty algebra and let B be non-empty. Change of base along the unique map

$$0 \longrightarrow B$$

sends every point over B to the unique point over 0.

DAMON. Because the fibre of points over the empty algebra is trivial.

CLEON. Yes. If this change-of-base functor were conservative, every morphism of points over B whose pullback is an isomorphism would already be an isomorphism.

NICIAS. Which morphism of points should we test?

CLEON. Take any point

$$p\colon A \to B, \qquad s\colon B \to A, \qquad ps = 1_B.$$

There is a morphism of points

$$(s, 1_B)\colon (1_B\colon B \to B) \longrightarrow (p\colon A \to B).$$

After pulling back along $0 \to B$, it becomes an isomorphism. Hence, by conservativity, it must already be an isomorphism.

THEON. So every split epimorphism over B is an isomorphism.

CLEON. Exactly. Now apply this to the product projection

$$A \times A \longrightarrow A,$$

which is split by the diagonal. It follows that $A \times A \to A$ is an isomorphism for every algebra A.

NICIAS. And in a concrete algebraic category this forces every algebra to have at most one element.

CLEON. Thus the protomodular case with empty initial object is necessarily degenerate.

DAMON. That is the point, Cleon. The term characterisation is not needed for this conclusion. It gives, in the varietal case, a useful reflection of the same phenomenon in terms of operations and identities; and there one must remember to read the case $n = 0$ separately. But your argument is broader: it rules out the non-trivial empty-initial case for prevarieties, and therefore also for varieties.

Cleon. Then, when empty algebras are allowed, there is no positive non-trivial example, either for varieties or for prevarieties.

Damon. Exactly. We must therefore pass to the other convention. Suppose that algebras and subalgebras are required to be non-empty. If finite completeness is imposed, the answer is still negative.

Theon. Indeed, a prevariety has small products.

Cleon. But products alone do not give an initial object.

Theon. No. We also need a small weakly initial family. Under the non-empty convention there is one: take one representative of each isomorphism class of one-generated algebras. Since every non-empty algebra contains a one-generated subalgebra, this family is weakly initial.

Nicias. And finite completeness supplies the missing equalisers.

Theon. Exactly. A prevariety has small products, and finite completeness supplies ordinary equalisers. Small products together with ordinary equalisers give all small limits. Hence the category is complete. Freyd's initial-object theorem[4] then gives an initial object.

Damon. Thus finite completeness rules out the desired phenomenon also under the non-empty convention. If an example exists, it must live where empty algebras are excluded and finite completeness is not imposed.

Cleon. And what provides such an example?

Nicias. Heaps, in the classical sense going back to Baer[5].

Cleon. Groups without a neutral element?

Nicias. Precisely. A heap is an algebra H with one ternary operation

$$[x, y, z]$$

satisfying

$$[x, y, y] = x, \qquad [y, y, x] = x,$$

and

$$[[x, y, z], u, v] = [x, y, [z, u, v]].$$

Every group gives a heap by

$$[x, y, z] = xy^{-1}z.$$

Conversely, once an element $a \in H$ is chosen, the formula

$$x \cdot_a y = [x, a, y]$$

turns H into a group with neutral element a.

Damon. Let us denote by $\mathbf{Heap}^+$ the category of non-empty heaps. The superscript reminds us that the empty heap is not being included.

Cleon. Why does $\mathbf{Heap}^+$ have no initial object?

[4] See [6, Theorem V.6.1].

[5] See [2].

NICIAS. Because constant maps between heaps are homomorphisms. If X is non-empty and H has two distinct elements a and b, then the two constant maps

$$X \longrightarrow H$$

with values a and b are distinct heap homomorphisms. Thus no object can have exactly one morphism into every heap.

THEON. And yet split pullbacks exist. If

$$p\colon E \to B, \qquad s\colon B \to E, \qquad ps = 1_B,$$

is a point and $u\colon X \to B$ is any heap homomorphism, then the set-theoretic pullback

$$X \times_B E$$

is a subheap of $X \times E$. It is non-empty, since for every $x \in X$ it contains

$$(x, s(u(x))).$$

CLEON. And why is $\mathbf{Heap}^+$ protomodular?

NICIAS. Choose $x_0 \in X$ and put $b_0 = u(x_0)$. These choices turn X and B into groups X_{x_0} and B_{b_0}, and u becomes a group homomorphism.

Now take a point over B,

$$p\colon E \to B, \qquad s\colon B \to E.$$

Using $s(b_0)$ as base point, E becomes a group, and p and s become group homomorphisms. Moreover, a morphism of points $f\colon E \to E'$ over B satisfies $fs = s'$, and therefore sends the chosen base point $s(b_0)$ to $s'(b_0)$. Thus it becomes a group homomorphism between the associated groups. This gives an equivalence

$$\mathrm{Pt}_B(\mathbf{Heap}^+) \simeq \mathrm{Pt}_{B_{b_0}}(\mathbf{Grp}).$$

Similarly,

$$\mathrm{Pt}_X(\mathbf{Heap}^+) \simeq \mathrm{Pt}_{X_{x_0}}(\mathbf{Grp}).$$

Under these equivalences, change of base along u is the ordinary change-of-base functor for groups:

$$u^*\colon \mathrm{Pt}_{B_{b_0}}(\mathbf{Grp}) \longrightarrow \mathrm{Pt}_{X_{x_0}}(\mathbf{Grp}).$$

Since groups are protomodular, this functor is conservative. Hence $\mathbf{Heap}^+$ is protomodular.

THEON. I recognise this argument. It is the concrete heap version of Bourn's result on associative Mal'cev operations with global support [4].

CLEON. So, under the non-empty convention, the answer is positive.

DAMON. Exactly. $\mathbf{Heap}^+$ is a non-trivial protomodular variety, in the non-empty reading, and hence also a prevariety, with no initial object. Indeed, under that convention, it is the equational class determined by the heap identities, with algebras and subalgebras understood to be non-empty.

THEON. But now add the empty heap.

Cleon. Then it becomes initial.

Theon. Exactly. But the resulting non-trivial category is no longer protomodular. Pull back points along the unique map

$$\varnothing \longrightarrow B$$

to a non-empty heap B. Every point over B is sent to the unique point over $\varnothing$, and every morphism of points is sent to the unique morphism there. If this change-of-base functor were conservative, all morphisms of points over B would have to be isomorphisms. This is false as soon as there is a non-trivial heap.

Damon. Thus the empty heap solves the initial-object problem only by changing the protomodularity problem.

Nicias. That is the curious feature of the example. The two categories differ only by one object, the empty heap. Yet removing that object destroys the initial object while preserving the behaviour of points; adding it back restores the initial object but destroys non-degenerate protomodularity.

Damon. So the same equational theory sits on both sides of the table. The difference is not the operation, but the convention on the empty algebra.

Cleon. So there are four questions hidden in the one I brought: empty algebras or non-empty algebras; varieties or prevarieties.

Damon. At least four. And if finite completeness is also kept separate, there are eight formulations rather than one.

Cleon. Then we have not answered the question by saying simply yes or no.

Damon. Indeed. This is what makes Aglianò's question a good one: it brings to the surface several conventions which are often left implicit. We have answered it by learning what the question means.

Theon. When the empty algebra is present, the initial object is already there. When finite completeness is imposed, it is forced. Only between these two constraints does the example survive.

Nicias. And there the example is almost invisible: take heaps, and remove just one object... the empty one!

Cleon. So the whole issue turns on the empty heap.

Damon. On the empty heap, and on the convention which decides whether it belongs to the category.

Cleon. Then the answer is not a single word, but a distinction.

Damon. Precisely.

References

[1] P. Aglianò, *Ideals in universal algebra: a survey*, SSAOS lecture notes, 2025.

[2] R. Baer, Zur Einordnung der Theorie der Gruppen in die der allgemeinen Verknüpfungssysteme, *Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse* (1929), 3–13.

[3] D. Bourn, *Normalization equivalence, kernel equivalence, and affine categories*, in A. Carboni, M. C. Pedicchio and G. Rosolini (eds.), *Category Theory, Como 1990*, Lecture Notes in Mathematics, vol. 1488, Springer, Berlin, 1991, pp. 43–62.

[4] D. Bourn, Baer sums and fibered aspects of Mal'cev operations, *Cahiers Topologie Géom. Différentielle Catég.* **40** (1999), no. 4, 297–316.

[5] D. Bourn and G. Janelidze, Characterization of protomodular varieties of universal algebras, *Theory Appl. Categ.* **11** (2003), 143–147.

[6] S. MacLane, *Categories for the working mathematician*, second ed., Springer-Verlag (1998).