# Extension theory and the calculus of butterflies

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**Abstract**

This paper provides a unified treatment of two distinct viewpoints concerning the classification of group extensions: the first uses weak monoidal functors, the second classifies extensions by means of suitable $H^2$-actions. We develop our theory formally, by making explicit a connection between (non abelian) $G$-torsors and fibrations. Then we apply our general framework to the classification of extensions in a semi-abelian context, by means of butterflies [1] between internal crossed modules. As a main result, we get an internal version of Dedecker's theorem on the classification of extensions of a group by a crossed module. In the semi-abelian context, Bourn's intrinsic Schreier-Mac Lane extension theorem [13] turns out to be an instance of our Theorem 6.3. Actually, even just in the case of groups, our approach reveals a result slightly more general than classical Schreier-Mac Lane theorem.

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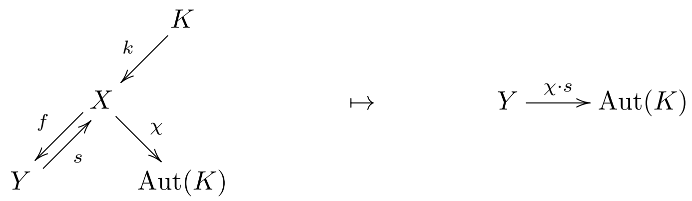
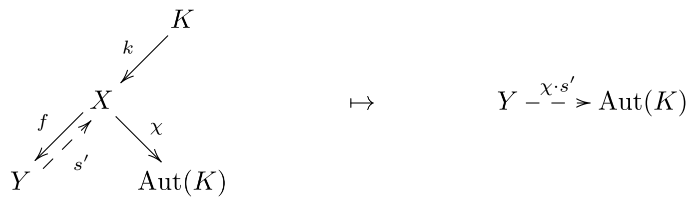
## 1 Introduction

Let $K$ and $Y$ be groups. It is well known that the set of (equivalence classes of) split extensions of $Y$ by $K$ is in bijection with the set of $Y$-actions on $K$. One way of realizing this bijection consists in considering a homomorphic section $s$ of $f$, and then composing with the canonical conjugation action of $X$ on its normal subgroup $K$, denoted by $\chi$ in the diagram below:




When the extension $K \xrightarrow{k} X \xrightarrow{f} Y$ is no longer split, the homomorphism $s$ fails to exist. Still, since $f$ is surjective, one can find a set-theoretical section $s'$ of $f$, and consider the composite $\chi \cdot s'$:




However, in this case $\chi \cdot s'$ is no longer an action, in general.

The group $\mathrm{Aut}(K)$ determines the internal groupoid in **Gp**

$$\mathrm{AUT}(K) \qquad = \qquad \begin{array}{c} K \rtimes \mathrm{Aut}(K) \\ d \big\downarrow \big\uparrow \big\downarrow c \\ \mathrm{Aut}(K) \end{array}$$

and the map $\chi \cdot s'$ underlies a (possibly weak) monoidal functor

$$D(Y) \to \mathrm{AUT}(K)\,,$$

where $D(Y)$ is the discrete internal groupoid associated with $Y$. In other words, $\chi \cdot s'$ is the object map of a functor between the underlying groupoids in **Set**. Notice that different choices of $s'$ give rise to different but isomorphic monoidal functors. This way, we extend the equivalence between split extensions and actions

$$\mathrm{SPLEXT}(Y, K) \simeq \mathbf{Gp}(Y, \mathrm{Aut}(K))$$

to the equivalence

$$\mathrm{EXT}(Y, K) \simeq \mathbf{2Gp}(D(Y), \mathrm{AUT}(K))\,,$$

where **2Gp** denotes the 2-category of groupoids in **Gp**, monoidal functors, and monoidal transformations. In this fashion, the map $\chi \cdot s'$ can be considered as a sort of *weak action*, a notion introduced by Blanco, Bullejos and Faro in [5].

Now, let us turn our attention to the *structure* of the set of equivalence classes of extensions $\mathrm{Ext}(Y, K)$, which is described by the classical Schreier-Mac Lane extension theory.

According to Mac Lane, an *abstract kernel* $\phi\colon Y \to \mathrm{Out}(K)$ is associated with any extension $(k, f)$, where $\mathrm{Out}(K) \cong \mathrm{Aut}(K)/\mathrm{Inn}(K)$ is the group of the outer homomorphisms of $K$, and $\phi$ induces a $Y$-action $\overline{\phi}$ on the center $Z(K)$ of the group $K$. Moreover, we have the following result, which merges Theorems 8.7 and 8.8 in [31, Chapter IV].

**Theorem 1.1** (Schreier-Mac Lane)**.** *Given a morphism* $\phi\colon Y \to \mathrm{Out}(K)$*, let* $\mathrm{OpExt}(Y, K, \phi)$ *be the set of extensions inducing the abstract kernel* $\phi$*.*

- $\mathrm{OpExt}(Y, K, \phi) \neq \emptyset$ *if, and only if,* $[\phi] = 0$*, where* $[\phi] \in H^3_{\overline{\phi}}(Y, Z(K))$ *is uniquely determined by* $\phi$*;*
- *If* $\mathrm{OpExt}(Y, K, \phi) \neq \emptyset$*, then* $H^2(Y, Z(K), \overline{\phi})$ *operates simply and transitively on* $\mathrm{OpExt}(Y, K, \phi)$*, so that there is a bijection* $\mathrm{OpExt}(Y, K, \phi) \cong H^2(Y, Z(K), \overline{\phi})$*.*

The Schreier-Mac Lane theorem gives a description of the set of (equivalence classes of) extensions as indexed by abstract kernels:

$$\mathrm{Ext}(Y, K) = \coprod_{\phi} \mathrm{OpExt}(Y, K, \phi)\,.$$

So far, we recalled two rather different points of view:

- extensions are classified by (some specific) weak monoidal functors,
- extensions are classified by $H^2$-actions.

The aim of this article is to show how the theory of classification of extensions is an instance of a more general one where these two viewpoints converge.

An intrinsic version of Schreier-Mac Lane theory has been introduced by Bourn in [14] for exact action representative categories. In this article, Bourn presents a cohomological classification of extensions by considering suitable groupoids of pretorsors (see also [13]) that give an internal description of the $H^2(Y, Z(K), \overline{\phi})$-actions.

We will extend Bourn's result to the wider class of extensions of a (discrete) object by a crossed module. This subject has been studied, in the case of group extensions by Dedecker ([23], see also [37]). Actually, in semi-abelian action representative or action accessible categories (see [9, 17]), Bourn's result is a consequence of Theorem 6.3 (see Corollary 6.5).

Our main tool will be internal butterflies. Butterflies have been introduced by Noohi in [37] in order to deal with monoidal functors between 2-groups (see also [2]). An intrinsic version for semi-abelian contexts has been developed in [1], where the authors show that butterflies are the morphisms in the bicategory of fractions of crossed modules, with respect to weak equivalences, or equivalently, of internal groupoids with respect to internal weak equivalences (i.e. internal functors which are fully faithful and essentially surjective on objects).

Indeed, in [34] the authors show that butterflies are the crossed module counterpart of a specific class of internal profunctors (where the last are the internal version of Bénabou's distributors [4]). The algebra of internal profunctors is a remarkable ingredient in Bourn's theory of extensions, nonetheless, the present work shows that in order to effectively perform computations, it is convenient to switch to butterflies. Our approach gives a recipe to deal with the theory of extensions in varieties of semi-abelian algebras where a simple notion of crossed module is defined, such as, for instance, groups, Lie-algebras over a field, rings and all *categories of interest* in the sense of Orzech [38].

A reason for this fact is the point of view according to which crossed modules generalize extensions since they generalize normal monomorphisms. For extensions with abelian kernel, it turns out that this generalization reduces the Baer sum of extensions to butterflies composition. A similar phenomenon can be described for the $H^2$-torsor structure of the set of equivalence classes of general extensions, thus making the calculus of butterflies a natural tool in the theory of extensions.

The paper is organized as follows. In Section 2, we present some basic results about torsors and fibrations. These will serve as a general framework from which the classification of extensions will follow in a purely formal way. In Section 3 we recall the needed results and properties about internal crossed modules in a semi-abelian category $\mathcal{C}$. Section 4 is devoted to the description of the bicategory of butterflies in a semi-abelian context. Section 5 is preparatory for the last one and provides the link between the general framework of Section 2 and the problem of classification of extensions, which is faced in Section 6.

# 2 The general framework

In this section we recall some basic notions from the theory of torsors, and we show how a fairly general situation produces canonically a setting that contextualizes the classification of extensions.

Let us consider the following elementary result.

**Lemma 2.1.** *Let $\mathcal{G}$ be a groupoid, and $x, y$ be objects of $\mathcal{G}$. Then:*

*(i) either $\mathcal{G}(x,y)$ is empty, or arrow composition in $\mathcal{G}$*

$$\mathcal{G}(x,y) \times \mathcal{G}(x,x) \to \mathcal{G}(x,y)$$

*defines a simply transitive (right) action $*$ of the group $\mathcal{G}(x,x)$ on the set $\mathcal{G}(x,y)$;*

*(ii) if $G$ is a subgroup of $\mathcal{G}(x,x)$, and $f$ is in $\mathcal{G}(x,y)$, the action above restricts to a simply transitive action*

$$fG \times G \to fG$$

*where $fG$ is the orbit $\{f * g \mid g \in G\}$.*

In [14], Bourn presents a cohomological classification of extensions by considering suitable groupoids of pretorsors (see also [13]). The starting point of the present work is the observation that Bourn's result, and consequently the classical Schreier-Mac Lane theorem on the classification of extensions, can be viewed as an instance of the very general Lemma 2.1 above.

## 2.1 $G$-torsors and fibrations

**Definition 2.2.** *Let $G$ be a group. A (right) $G$-*torsor *is a non-empty set $S$ equipped with a* simply transitive *(right) $G$-action $*$.*

This means that the action is such that only the identity element of the group acts trivially and there is only one orbit. One can easily see that a $G$-set $S$ is a $G$-torsor if and only if the assignment $(x, g) \mapsto (x, x * g)$ establishes a bijection between the sets $S \times G$ and $S \times S$.

With this characterization in mind, one can state an internal notion of $G$-torsor. Notice that, in the internal case, the non-emptiness request translates into the condition that $S$ is *inhabited* (i.e. with a strong epimorphism to the terminal object).

**Definition 2.3.** *Let $\mathcal{C}$ be a category with finite products and $G$ a group object in $\mathcal{C}$. A (right) $G$-*torsor in $\mathcal{C}$ *is an inhabited object $S$ equipped with a (right) $G$-action $\xi \colon S \times G \to S$, such that the following map is an isomorphism:*

$$\langle \pi_1^{S\times G}, \xi \rangle \colon S \times G \to S \times S\,.$$

For instance, one can choose a set $B$ and define $\mathfrak{G}$-torsors in $\mathbf{Set} \downarrow B$, where $\mathfrak{G}$ is a group in $\mathbf{Set} \downarrow B$. In particular one can choose a group $G$, and observe that

$$\pi_2^{G\times B} \colon G \times B \to B$$

inherits from $G$ a group structure in $\mathbf{Set} \downarrow B$.

**Definition 2.4.** *Let $G$ be a group and $B$ a set. A (right) $G$-*torsor over $B$ *is a $\pi_2^{G\times B}$-torsor in $\mathbf{Set} \downarrow B$.*

Notice that this definition can be stated in any category with finite products. It's worth to make it more explicit in terms of the actions of $G$ in **Set**: indeed, one can easily prove that an object $p \colon S \to B$ in $\mathbf{Set} \downarrow B$ is a $G$-torsor over $B$ precisely when, for each $b \in B$, $p^{-1}(b) \subset S$ is a $G$-torsor. In other words, $G$ acts on $S$ fiberwise and the action is simply transitive on each fiber.

Such a situation appears naturally when we consider a functor

$$\Pi \colon \mathcal{G} \to \mathcal{M}\,,$$

where $\mathcal{G}$ is a groupoid and $\mathcal{M}$ is a category.

Let us fix two objects $x$ and $y$ in $\mathcal{G}$. For an arrow $\alpha \colon \Pi(x) \to \Pi(y)$, we shall denote by $\mathcal{G}_\alpha(x, y)$ the subset of $\mathcal{G}(x, y)$ of all those arrows $f \colon x \to y$ such that

$\Pi(f) = \alpha$. If moreover $\Pi(x) = \Pi(y)$ and $\alpha = 1_{\Pi(x)}$, we shall adopt the simpler notation $\mathcal{G}_1(x, y)$.

As pointed out in Lemma 2.1 ($i$), if the hom-set $\mathcal{G}(x, y)$ is not empty, it inherits a structure of $\mathcal{G}(x, x)$-torsor from the arrow composition in $\mathcal{G}$. Here we identify the groupoid with one object $\mathcal{G}(x, x)$ with the corresponding group of its arrows, composition law given again by the groupoid structure of $\mathcal{G}$.

Furthermore, applying Lemma 2.1 ($ii$) to the group $G = \mathcal{G}_1(x, x)$, we get immediately the following result.

**Proposition 2.5.** *Let $B = \Pi(\mathcal{G}(x, y)) \subset \mathcal{M}(\Pi(x), \Pi(y))$ and $G$ as above. Then*

*(i) either $\mathcal{G}(x, y)$ is empty, or it is a $G$-torsor over $B$;*

*or, equivalently,*

*(ii) for each $\phi \in \mathcal{M}(\Pi(x), \Pi(y))$, either $\mathcal{G}_\phi(x, y)$ is empty, or it is a $G$-torsor.*

*Proof.* ($i$) is equivalent to ($ii$) as noticed after Definition 2.4.

To prove ($ii$), it suffices to observe that, for any $f : x \to y$ such that $\Pi(f) = \phi$, $\mathcal{G}_\phi(x, y)$ is nothing but the orbit $fG$. Then the thesis follows from point ($ii$) of Lemma 2.1. □

We can somehow relax the hypothesis that $\mathcal{G}$ is a groupoid, if we ask for some conditions to hold for the functor $\Pi$.

Let us recall that a fibration

$$\Pi : \mathcal{E} \to \mathcal{M}$$

is a functor with enough cartesian liftings, i.e. $\Pi$ is such that, for every object $y$ in $\mathcal{E}$, and for every arrow $\phi$ with codomain $\Pi(y)$, there exists a cartesian arrow $d_\phi : x' \to y$ such that $\Pi(d_\phi) = \phi$.

It is worth to point out that cartesian arrows can be characterized according to the following statement (we follow here the terminology used by Borceux in [6], although some authors call *hypercartesian* what we call a cartesian arrow).

**Lemma 2.6.** *Let $\Pi : \mathcal{E} \to \mathcal{M}$ be a functor. An arrow $d_\phi : x' \to y$ is cartesian over $\phi = \Pi(d_\phi)$ if and only if, for every $z$ in $\mathcal{E}$ and every $\alpha : \Pi(z) \to \Pi(x')$, the map*

$$d_\phi \cdot - : \mathcal{E}_\alpha(z, x') \to \mathcal{E}_{\phi \cdot \alpha}(z, y)$$

*is a bijection.*

We recall also that a morphism of fibrations is a commutative square of functors

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & \mathcal{E}' \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \Pi'} \\ \mathcal{M} & \xrightarrow[G]{} & \mathcal{M}' \end{array}$$

such that $\Pi$ and $\Pi'$ are fibrations and $F$ preserves cartesian arrows.

**Proposition 2.7.** *Let us consider a fibration of categories $\Pi\colon \mathcal{E} \to \mathcal{M}$, such that the fibers of $\Pi$ are groupoids. Let us fix two objects $x$ and $y$ of $\mathcal{E}$, and define*

*the set $B = \Pi(\mathcal{E}(x, y)) \subset \mathcal{M}(\Pi(x), \Pi(y))$,*

*the group $G = \mathcal{E}_1(x, x)$.*

*Then*

*(i) either $\mathcal{E}(x, y)$ is empty or it is a $G$-torsor over $B$, the action given by arrow composition;*

*or, equivalently,*

*(ii) for each $\phi \in \mathcal{M}(\Pi(x), \Pi(y))$, either $\mathcal{E}_\phi(x, y)$ is empty, or it is a $G$-torsor.*

*Proof.* Again, it suffices to prove the second statement. To this end, let us consider an arrow $f$ in $\mathcal{E}_\phi(x, y)$, and a cartesian lifting $d_\phi\colon x' \to y$ of $\phi$ at $y$. By Lemma 2.6, we have a bijection

$$d_\phi \cdot -\colon \mathcal{E}_1(x, x') \to \mathcal{E}_\phi(x, y)$$

Now, it suffices to transport along this bijection the canonical $G$-torsor structure of $\mathcal{E}_1(x, x')$ described in Proposition 2.5. □

As we shall explain, for suitable choices of $\mathcal{E}$, $\mathcal{M}$ and $\Pi$, the classical Schreier-Mac Lane Theorem is obtained as an application of Proposition 2.7 (*ii*). Indeed, the functor $\Pi$ restricted to $\mathcal{E}(x, y)$ is specialized, for instance in the case of group extensions, to the assignment that associates with any extension:

$$K \xrightarrow{k} X \xrightarrow{f} Y \ ,$$

its abstract kernel $\phi$. Finally, the acting group $G$ is nothing but the usual second Mac Lane-cohomology group of $Y$ with coefficients in the center of $K$ (the Baer group).

Indeed, even just in the case of groups, the setting developed above yields a result slightly more general than the Schreier-Mac Lane theorem. This is a consequence of point ($i$) of Proposition 2.7, that allows to interpret the action of the Baer group as fibred over the set of those abstract kernels which admit a lifting.

**Remark 2.8.** Our general setting seems to be robust enough to sharpen the theory in order to deal with the classification of extensions (possibly with coefficients in a given crossed modules), and not just, as it is reported in the classical literature, with the classification of *equivalence classes* of extensions.

This can be done by considering the 2-dimensional analogue of the situation described above: we can start with a bigroupoid $\mathcal{G}$, so that $\mathcal{G}(x, y)$ is a weak categorical group, and the main action on the groupoid $\mathcal{G}(x, y)$ yields a categorical torsor, in the sense of [20]. Along these lines, one can further choose a bicategory $\mathcal{M}$ and a homomorphism of bicategories $\Pi\colon \mathcal{G} \to \mathcal{M}$.

Although we do not develop specifically this approach in the present work, it is worth to observe that, by the coherence theorems that involve bicategorical constructions, one can actually deal with the numerous details of the definition of categorical torsor in quite a straightforward way.

# 3 Internal crossed modules

The context where we develop our theory is that of semi-abelian categories [29], i.e. categories which are pointed, Barr-exact, protomodular and with finite coproducts. For a detailed account, the reader is referred to [7].

In addition, we require that the so-called "Smith is Huq" condition (SH) holds (see [35]). Examples of such categories are those of groups, rings, associative algebras, Lie algebras, Poisson algebras and, in general, any *category of interest* in the sense of Orzech [38] (see also [36]).

## 3.1 Actions

Semi-abelian categories are a convenient setting for working with *internal actions*. Here we briefly recall their definition from [9].

Let $\mathcal{C}$ be a semi-abelian category. Let

$$\mathbf{Pt}_A(\mathcal{C}) = 1_A \downarrow (\mathcal{C} \downarrow A)\,,$$

i.e. the category of points of $\mathcal{C} \downarrow A$. For every object $A$ of $\mathcal{C}$, the kernel functor $\mathrm{Ker}_A \colon \mathbf{Pt}_A(\mathcal{C}) \to \mathcal{C}$ is monadic. The corresponding monad is denoted by $A\flat(-)$, defined, for any object $X$ of $\mathcal{C}$, by the kernel diagram:

$$A\flat X \xrightarrow{\kappa_{A,X}} A + X \xrightarrow{[1,0]} A\,.$$

The $A\flat(-)$-algebras are called *internal $A$-actions* [9, 16]. The category $\mathcal{C}^{A\flat(-)}$ of algebras is denoted by $\mathbf{Act}(A,-)$. For an action $\xi \colon A\flat X \to X$, the semidirect product of $X$ with $A$, with action $\xi$ is the split epimorphism corresponding to $\xi$ via the canonical comparison $\Xi \colon \mathbf{Pt}_A(\mathcal{C}) \to \mathbf{Act}(A,-)$. It is computed explicitly (see [33]) via the coequalizer:

$$A\flat X \underset{i_X \cdot \xi}{\overset{\kappa_{A,X}}{\rightrightarrows}} A + X \xrightarrow{q_\xi} X \rtimes_\xi A\,.$$

Canonical examples of internal actions follow:

- the *trivial action* of $A$ on $X$ is given by the composition

$$\rho_{A,X} = \rho_X \colon\; A\flat X \xrightarrow{\kappa_{A,X}} A + X \xrightarrow{[0,1]} X \;\;;$$

- the *conjugation action* of $X$ is given by the composition

$$\chi_X \colon\; X\flat X \xrightarrow{\kappa_{X,X}} X + X \xrightarrow{[1,1]} X \;\;;$$

- for a kernel $K \to X$, the *conjugation action* of $X$ restricts to an action:

$$\chi_{X,K} \colon \; X \flat K \longrightarrow K \;.$$

Both $\rho_{A,X}$ and $\chi_X$ are natural in their variables.

Moreover, it is worth to observe that, for any action $\xi \colon B\flat X \to X$ and any morphism $f \colon A \to B$, the composite

$$f^*(\xi) \colon \; A\flat X \xrightarrow{f\flat 1_X} B\flat X \xrightarrow{\xi} X$$

defines an action, called the *pullback* action of $\xi$ along $f$ (indeed, the above composition amounts to a pullback via the canonical comparison $\Xi$).

## 3.2 Crossed modules

Internal pre-crossed and crossed modules in a semi-abelian category were defined by Janelidze in [28]. A pre-crossed module $\mathbb{G} = (\partial_G, \xi_G)$ in $\mathcal{C}$ is a map $\partial_G$ together with an action $\xi_G$

$$G_0\flat G \xrightarrow{\xi_G} G \xrightarrow{\partial_G} G_0$$

such that the following diagram commutes:

$$\begin{array}{ccc} G_0\flat G & \xrightarrow{\xi_G} & G \\ {\scriptstyle 1\flat\partial_G}\big\downarrow & & \big\downarrow{\scriptstyle \partial_G} \\ G_0\flat G_0 & \xrightarrow[\chi_{G_0}]{} & G_0 \end{array}$$

Whenever the condition (SH) holds (see [35]), a pre-crossed module is a crossed module if, in addition, the following diagram commutes:

$$\begin{array}{ccc} G\flat G & \xrightarrow{\chi_G} & G \\ {\scriptstyle \partial_G\flat 1}\big\downarrow & & \big\downarrow{\scriptstyle 1} \\ G_0\flat G & \xrightarrow[\xi_G]{} & G \end{array}$$

We shall refer to the commutativity of the first diagram as the *pre-crossed module condition* and to the commutativity of the second diagram as the *Peiffer condition.*

A morphism of pre-crossed modules $\mathbb{H} \to \mathbb{G}$ is a pair $(f, f_0)$ of maps making the following diagram commute:

$$\begin{array}{ccc}
H_0\flat H & \xrightarrow{f_0\flat f} & G_0\flat G \\
\downarrow{\scriptstyle \xi_H} & (i) & \downarrow{\scriptstyle \xi_G} \\
H & \xrightarrow{f} & G \\
\downarrow{\scriptstyle \partial_H} & (ii) & \downarrow{\scriptstyle \partial_G} \\
H_0 & \xrightarrow[f_0]{} & G_0
\end{array} \qquad (1)$$

We will refer to the commutativity of $(i)$ above by saying that the pair $(f, f_0)$ is equivariant with respect to the actions involved. A morphism of crossed modules is just a morphism of the underlying pre-crossed modules.

In [28] it is proved that the category $\mathbf{Gpd}(\mathcal{C})$ of internal groupoids and internal functors is equivalent to the category $\mathbf{XMod}(\mathcal{C})$ of internal crossed modules, and in [1] this equivalence is extended to a biequivalence of bicategories.

The process of associating a crossed module with a groupoid is described explicitly in [34]. Here we merely fix the notation. Given a groupoid $\mathbb{G}$:

$$G_1 \underset{d}{\overset{c}{\rightrightarrows}} G_0 \; (u\colon G_0 \to G_1),$$

its associated crossed module is

$$G_0\flat G \xrightarrow{\xi_G} G \xrightarrow{\partial_G = c\cdot h} G_0 \, ,$$

where $h = \ker(d)$. We will often use the same notation for the groupoid $\mathbb{G}$ and the crossed modules associated with it, unless this would cause confusion.

Many relevant notions concerning crossed modules arise by direct translation of the corresponding notions for groupoids. For instance, a *discrete fibration* $(f, f_0)$ between crossed modules is a morphism that, considered as a functor between the corresponding groupoids, is a discrete fibration. One can easily verify that this is the case if, and only if, $f$ is an isomorphism.

We recall now an interesting feature of crossed modules (also shared by pre-crossed modules): their underlying maps are proper, i.e. the image factorization of a crossed module map $\partial_G$ produces a normal monomorphism $m_G$:

$$G \overset{e_G}{\twoheadrightarrow} Im(\partial_G) \overset{m_G}{\rightarrowtriangle} G_0 \, , \qquad \partial_G = m_G \cdot e_G .$$

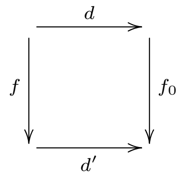
Indeed, it is possible to show that this construction underlies a morphism of crossed modules

$$(e_G, 1_{G_0})\colon \partial_G \to m_G \, .$$

Interpreted in terms of groupoids, this is nothing but the projection of $\mathbb{G}$ onto its support (i.e. the kernel pair of $\mathrm{coeq}(d, c)$).

The following result concerning proper maps has some consequences when the proper maps are crossed modules (compare with Lemma A.1 in [13]).

**Lemma 3.1.** *In a homological category $\mathcal{C}$, we consider two proper maps $d$ and $d'$, and a morphism $(f, f_0)$ between them:*

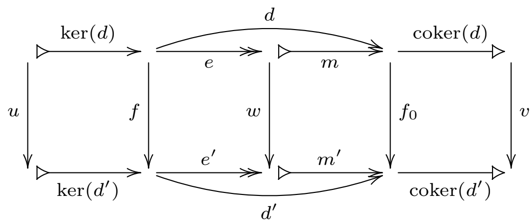



*Then the following properties hold:*

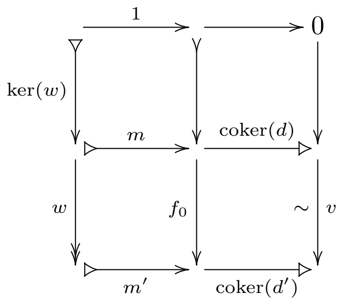
*(i) if $f$ and the corestriction to the cokernels of $d$ and $d'$ of $(f, f_0)$ are isomorphisms, then $f_0$ is a regular epimorphism;*

*(ii) if $f_0$ and the restriction to the kernels of $d$ and $d'$ of $(f, f_0)$ are isomorphisms, then $f$ is a normal monomorphism.*

*Proof.* Let us consider the following diagram, where $(e, m)$ and $(e', m')$ are the (regular epi, mono) factorizations of $d$ and $d'$ respectively, and the maps $u$, $v$ and $w$ are induced by the universal properties of kernels, cokernels and factorization:




$(i)$ Suppose that $f$ and $v$ are isomorphisms. Then $w$ is a regular epimorphism. Since the category is homological, we can apply Nine Lemma to the following commutative diagram:




thus proving that the central column is a short exact sequence, and in particular $f_0$ is a regular epimorphism. Notice also that the square $m'w = f_0 m$ is a pullback and a pushout.

($ii$) Assume now that $u$ and $f_0$ are isomorphisms. The property follows by applying a dual argument to the squares $e'f = we$ and $f\ker(d) = \ker(d')u$. $\square$

### 3.3 Homotopy invariants

If $(\partial_G, \xi_G)$ is a crossed module in groups, with $(\partial_G, \xi_G)$ it is possible to associate in a natural way the groups:

$$\pi_0(\partial_G, \xi_G) = \mathrm{Coker}(\partial_G)\,, \qquad \pi_1(\partial_G, \xi_G) = \mathrm{Ker}(\partial_G)\,. \tag{2}$$

Moreover, the action $\xi_G$ induces a $\pi_0(\partial_G, \xi_G)$-module structure on $\pi_1(\partial_G, \xi_G)$, and this process gives rise to a functor

$$\pi_{0,1} \colon \mathbf{XMod}(\mathbf{Gp}) \to \mathbf{Mod}(\mathbf{Gp})\,,$$

where the codomain is the full subcategory of $\mathbf{XMod}(\mathbf{Gp})$ whose objects are crossed modules whose underlying map in $\mathbf{Gp}$ is trivial. Namely, an object in $\mathbf{Mod}(\mathbf{Gp})$ is nothing but the zero map $0 \colon A \to Y$ between a $Y$-module $A$ and the acting group $Y$, together with the given action.

The aim of the present section is to give an analogue of this construction in the intrinsic setting.

Let us be given a crossed module $(\partial_G, \xi_G)$ in a semi-abelian category $\mathcal{C}$, and its associated internal groupoid $\mathbb{G}$. Let $\pi_0$ and $\pi_1$ be defined as in (2). It is not difficult to see that $\pi_0(\partial_G, \xi_G) = \pi_0(\mathbb{G})$, i.e. the coequalizer of the pair $(d, c)$, domain and codomain maps of $\mathbb{G}$, while $\pi_1(\partial_G, \xi_G) = \pi_1(\mathbb{G})$, i.e. the intersection $\mathrm{Ker}(c) \cap \mathrm{Ker}(d)$.

In [13, Definition 1.5], the author defines a *global direction* functor. In our setting, the global direction of a groupoid $\mathbb{G}$ is the totally disconnected groupoid produced on the right hand side by the following pushout of solid arrows:

$$\begin{array}{ccc}
R[\langle d,c\rangle] & \longrightarrow & d_1(\mathbb{G}) \\
{\scriptstyle r_1}\downarrow\uparrow\downarrow{\scriptstyle r_2} & & \uparrow\downarrow \\
G_1 & \longrightarrow & \pi_0(\mathbb{G}) \\
\downarrow{\scriptstyle \langle d,c\rangle} & & \\
G_0 \times G_0 & &
\end{array} \tag{3}$$

where $G_1 = G \rtimes G_0$. Bourn also shows that the two downward directed squares are pullbacks. This gives us a discrete fibration of groupoids, which corresponds to the following morphism of crossed modules:

$$\begin{array}{ccc}
\pi_1(\mathbb{G}) & \xrightarrow{1} & \pi_1(\mathbb{G}) \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle 0} \\
G_1 & \longrightarrow & \pi_0(\mathbb{G})
\end{array}$$

where $i$ is the normal inclusion. Indeed, $\mathrm{Ker}(\langle d, c\rangle) = \pi_1(\mathbb{G}) = \mathrm{Ker}(c) \cap \mathrm{Ker}(d)$, and this yields a $\pi_0(\mathbb{G})$ module structure on $\pi_1(\mathbb{G})$. Let us observe that the equivariance condition here means precisely that the action of $\pi_0(\mathbb{G})$ on $\pi_1(\mathbb{G})$ is induced by $\xi_G$. The previous discussion is summarized by the following result.

**Lemma 3.2.** *Let $\mathcal{C}$ be a semi-abelian category, and let $\mathbb{G}$ be an internal groupoid in $\mathcal{C}$. The global direction of $\mathbb{G}$ is the totally disconnected groupoid corresponding to the $\pi_0(\mathbb{G})$-module structure on $\pi_1(\mathbb{G})$.*

A morphism of crossed modules (an internal functor) is called *weak equivalence* if it induces isomorphisms on $\pi_0$ and $\pi_1$. Indeed, these correspond precisely to fully faithful and essentially surjective internal functors (see [24]).

## 3.4 Two factorization systems

In what follows, we will need two factorization systems which have been developed for internal groupoids (see [10, 19]). The first one is the factorization system canonically associated with the fibration $\pi_0$. The second one, (final, discrete fibration), was introduced for general categories in [40]. In the case of internal groupoids final functors are conveniently defined as the orthogonal class with respect to discrete fibrations. We recall from [21] their translation for internal crossed modules.

### The ($\pi_0$-invertible, $\pi_0$-cartesian) factorization

Let us consider a morphism of internal crossed modules

$$\underline{f} = (f, f_0) \colon \mathbb{H} \to \mathbb{G}$$

We proceed as follows.

$$\begin{array}{ccccc}
H & \xrightarrow{\ f\ } & G & \xrightarrow{\ 1\ } & G \\
{\scriptstyle \partial_H}\downarrow & & \downarrow{\scriptstyle \partial'} & & \downarrow{\scriptstyle \partial_G} \\
H_0 & \xrightarrow{\ f'_0\ } & G'_0 & \xrightarrow{\ f''_0\ } & G_0 \\
{\scriptstyle \mathrm{coker}(\partial_H)}\downarrow & & \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{coker}(\partial_G)} \\
\pi_0(\mathbb{H}) & \xrightarrow[\ 1\ ]{} & \pi_0(\mathbb{H}) & \xrightarrow[\ \pi_0(\underline{f})\ ]{} & \pi_0(\mathbb{G})
\end{array} \qquad (4)$$

First we compute the bottom right pullback, and construct the comparison maps $f'_0 = \langle \mathrm{coker}(\partial_H), f_0\rangle$ and $\partial' = \langle 0, \partial_G\rangle$. It can be shown that $\partial'$ inherits a crossed module structure from that of $\mathbb{G}$, in a way such that the two upper squares are crossed module morphisms. Finally, since crossed modules are proper morphisms, the vertical unlabelled arrow is the cokernel of $\partial'$, so that the two upper morphisms form the desired factorization.

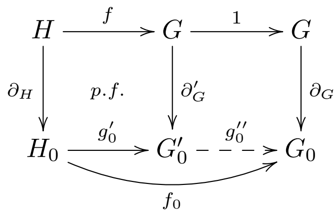
**Remark 3.3.** Let us observe that the morphism $(1_G, f_0'')$ produced on the right hand side is not only a discrete fibration, but it is a cartesian map with respect to the functor $\pi_0$. Moreover, the construction above explains that $\pi_0$-cartesian maps induce isomorphisms on $\pi_1$ (it suffices to observe that $\partial'$ and $\partial_{\mathbb{G}}$ are proper maps with isomorphic images).

**The (final, discrete fibration) factorization**

Again, let us consider the morphism of internal crossed modules

$$\underline{f} = (f, f_0) \colon \mathbb{H} \to \mathbb{G}$$

As explained in [21], a *push forward* construction for crossed modules is available, under suitable hypothesis, in any semi-abelian category. In particular, given a morphism $(f, f_0)$, we can compute the push forward $\mathbb{G}' = (\partial_G', \xi_G')$ of $\mathbb{H} = (\partial_H, \xi_H)$ along the map $f$.




By the universal property of the push forward, we get the factorization $f_0 = g_0'' \cdot g_0'$, that induces the factorization of $\underline{f}$ in the diagram. In particular, the morphism $(f, g_0')$ is final, while the morphism $(1_G, g_0'')$ is a discrete fibration. We recall from [21] that the following characterization holds:

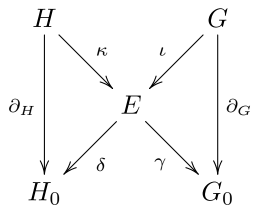
**Proposition 3.4.** *A morphism of crossed modules $\underline{f}$ is final if and only if $\pi_0(\underline{f})$ is an isomorphism, and, $\pi_1(\underline{f})$ is a regular epimorphism.*

# 4 Internal butterflies

Internal butterflies have been introduced in [1] in order to describe a notion of weak map between crossed modules in a semi-abelian category. The interested reader is referred to [1] for the notations and basic results. The groupoidal version of butterflies are called *fractors* in [34], and they are a special kind of internal profunctors between internal groupoids. In the case of groups, fractors correspond to monoidal functors, i.e. functors between the underlying groupoids in **Set** that preserve the group structure up to (coherent) isomorphism. More generally, also in the case of rings and of Lie algebras, fractors represent functors weakly preserving the algebraic structure, so that we can legitimately recognize fractors (respectively butterflies) as weak maps between internal groupoids (respectively crossed modules). A further evidence of this is the fact that the inclusion of the 2-category of groupoids in the bicategory of fractors is the bicategorical localization of the first with respect to weak equivalences.

## 4.1 The bicategory of butterflies

**Definition 4.1.** *Let $\mathcal{C}$ be a semi-abelian category satisfying* (SH)*, and consider two internal crossed modules $\mathbb{H} = (\partial_H, \xi_H)$ and $\mathbb{G} = (\partial_G, \xi_G)$. A* butterfly $\widehat{E} \colon \mathbb{H} \nrightarrow \mathbb{G}$ *is a commutative diagram of the form*




(5)

*such that*

*i.* $(\kappa, \gamma)$ *is a complex, i.e.* $\gamma \cdot \kappa = 0$*,*

*ii.* $(\iota, \delta)$ *is a short exact sequence, i.e.* $\delta = \operatorname{coker}\iota$ *and* $\iota = \ker \delta$*,*

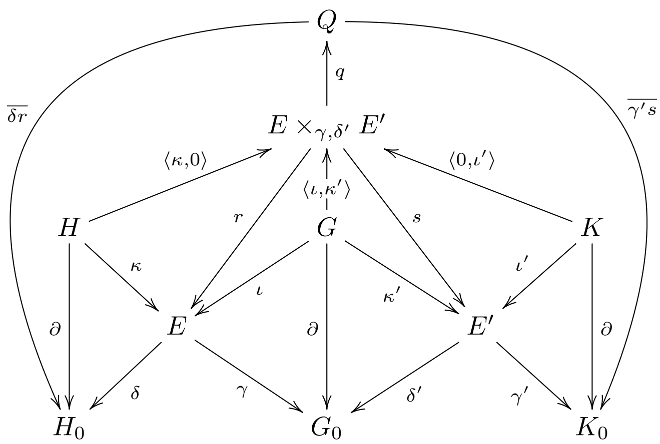
*iii. The action of $E$ on $H$ induced by that of $H_0$ on $H$ via $\delta$ makes $\kappa \colon H \to E$ a pre-crossed module,*

*iv. The action of $E$ on $G$ induced by that of $G_0$ on $G$ via $\gamma$ makes $\iota \colon G \to E$ a pre-crossed module.*

*A morphism of butterflies $\widehat{E}, \widehat{E}' \colon \mathbb{H} \nrightarrow \mathbb{G}$ is an arrow $\alpha \colon E \to E'$ commuting with the $\kappa$'s, the $\iota$'s, the $\delta$'s and the $\gamma$'s.*

It is easy to see that the definition of butterfly implies that $\kappa$ and $\iota$ are indeed crossed modules, and that the definition of morphism of butterflies implies that $\alpha$ is an isomorphism.

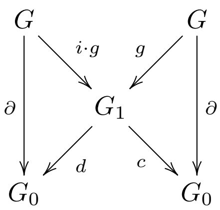
In order to obtain a bicategory $\mathbf{Bfly}(\mathcal{C})$ of crossed modules and butterflies, we describe now the composition of butterflies. Let us consider two butterflies $\widehat{E} \colon \mathbb{H} \nrightarrow \mathbb{G}$ and $\widehat{E}' \colon \mathbb{G} \nrightarrow \mathbb{K}$. The composite $\widehat{E}' \cdot \widehat{E} \colon \mathbb{H} \nrightarrow \mathbb{K}$ is defined by the following construction:




(6)

where $E \times_{\gamma,\delta'} E'$ is the pullback of $\gamma$ and $\delta'$, and $Q$ is the cokernel of $\langle \iota, \kappa' \rangle$. The morphisms that give the structure of the composite $\widehat{E}' \cdot \widehat{E}$ are $q \cdot \langle \kappa, 0 \rangle$, $q \cdot \langle 0, \iota' \rangle$, and $\overline{\delta r}$, $\overline{\gamma' s}$, obtained by the universal properties of the (co)limits involved.

For each crossed module $\mathbb{G} = (\partial_G, \xi_G)$, its identity butterfly is given by the diagram




(7)

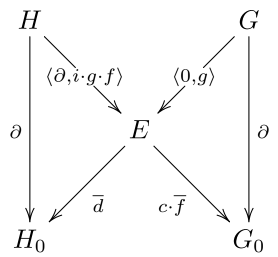
where $(G_1, G_0, d, c, e)$ is the groupoid associated with $(\partial_G, \xi_G)$, $i$ its inversion morphism and $g = \ker(d)$. Butterfly composition extends to 2-cells, and these data form a bicategory $\mathbf{Bfly}(\mathcal{C})$, which is locally groupoidal, i.e. hom-categories are groupoids.

The 2-category of crossed modules embeds in the bicategory of butterflies:

$$\mathcal{B}\colon \mathbf{XMod}(\mathcal{C}) \to \mathbf{Bfly}(\mathcal{C})\,. \tag{8}$$

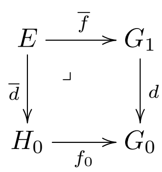
The homomorphism $\mathcal{B}$ is the identity on objects; for a morphism of crossed modules $\underline{f} = (f, f_0)\colon (\partial_H, \xi_H) \to (\partial_G, \xi_G)$, one defines:

$\mathcal{B}(\underline{f}) =$




,

where




is a pullback. The universal property of pullbacks determines the action of $\mathcal{B}$ on 2-arrows.

Consider that the regular epimorphism $\overline{d}$ is indeed a split epimorphism. It can be shown that any butterfly where the sequence $(\kappa, \delta)$ is split come from a (strict) morphism of crossed modules. Such butterflies are called *representable.*

**Remark 4.2.** If we remove the exactness condition from the diagonal, i.e. if we require the two diagonals to be just complexes, we get what is called a *crossed* *profunctor*, i.e. the crossed module version of an internal profunctor (see [1]). Crossed profunctors form a bicategory $\mathbf{XProf}(\mathcal{C})$ (compositions and identities as for butterflies), and one has an inclusion of bicategories

$$\mathbf{Bfly}(\mathcal{C}) \hookrightarrow \mathbf{XProf}(\mathcal{C}) \, .$$

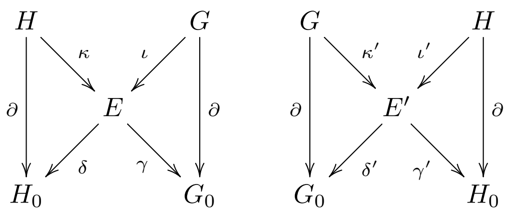
Let us observe that if we flip a butterfly $\widehat{E}$ horizontally, i.e. if we exchange domain and codomain of the butterfly, the outcome is no longer a butterfly, but it is still a crossed profunctor. This is denoted by $\widehat{E}^{\circ}$.

A special kind of butterfly is the class of *flippable* butterflies, i.e. those butterflies such that also the pair $(\kappa, \gamma)$ is short exact. It is not difficult to see that these are indeed equivalences in $\mathbf{Bfly}(\mathcal{C})$. In fact, we have more.

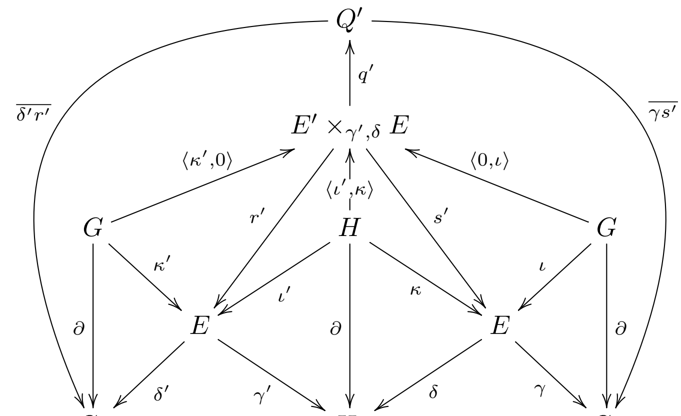
**Proposition 4.3.** *A butterfly* $\widehat{E} \colon \mathbb{H} \nrightarrow \mathbb{G}$ *is an equivalence in* $\mathbf{Bfly}(\mathcal{C})$ *if and only if it is flippable.*

*Proof.* The "if" part is Proposition 3.8 in [1], so it only remains to prove that every equivalence is a flippable butterfly.

Suppose that $\widehat{E} \colon \mathbb{H} \nrightarrow \mathbb{G}$ and $\widehat{E'} \colon \mathbb{G} \nrightarrow \mathbb{H}$ are quasi-inverse to each other:



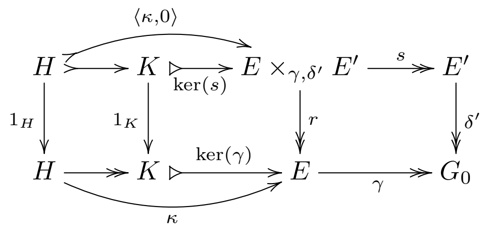

First of all, we prove that $\gamma = \mathrm{coker}(\kappa)$. Indeed, if we compute the composite $\widehat{E} \cdot \widehat{E'} \cong \mathrm{id}_{\mathbb{G}}$:




we have that the arrow $\overline{\gamma s'}$ is a regular epimorphism, being isomorphic to the codomain projection of the groupoid $\mathbb{G}$. Since $q'$ is a cokernel, then also $\overline{\gamma s'} \cdot q' =$

$\gamma \cdot s'$ is a regular epimorphism, and consequently so is $\gamma$. By symmetry, $\gamma'$ is also a regular epimorphism, hence so is its pullback $s'$. Then, having in mind that $\overline{\gamma s'}$ is the cokernel of $q' \cdot \langle \kappa', 0\rangle$ and $s' \cdot \langle \kappa', 0\rangle = 0$, it is easy to see that the square $\gamma \cdot s' = \overline{\gamma s'} \cdot q'$ is a pushout. As a consequence, since $q' = \mathrm{coker}(\langle \iota', \kappa\rangle)$, we have that $\gamma = \mathrm{coker}(s' \cdot \langle \iota', \kappa\rangle) = \mathrm{coker}(\kappa)$.

Now, if we consider the composite $\widehat{E'} \cdot \widehat{E} \cong \mathrm{id}_{\mathbb{H}}$ (look at diagram (6) replacing $\mathbb{K}$ with $\mathbb{H}$), we have that $q \cdot \langle k, 0\rangle$ is a monomorphism, being isomorphic to the kernel of the codomain projection of the groupoid $\mathbb{H}$. Then $\langle k, 0\rangle$ is also a monomorphism. Now, let us consider the following diagram:

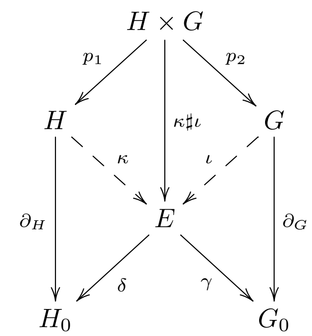



Since the right hand square is a pullback, $\ker(s) = \ker(\gamma)$. Then, since $s \cdot \langle \kappa, 0\rangle = 0$, there is a monomorphic comparison between $H$ and $K$. On the other hand, $\kappa$ is a pre-crossed module and $\gamma = \mathrm{coker}(\kappa)$, so that $\ker(\gamma)$ is the monomorphic part of the (regular epi, mono) factorization of $\kappa$. Hence, the comparison between $H$ and $K$ is also a regular epimorphism, thus an isomorphism. We have just proved that $(\kappa, \gamma)$ is a short exact sequence as desired. $\square$

We denote by $\mathbf{Bfly}_{eq}(\mathcal{C})$ the sub-bicategory (actually a sub-bigroupoid) of $\mathbf{Bfly}(\mathcal{C})$ formed by flippable butterflies. The corresponding profunctors are called *regularly fully faithful* in [14].

## 4.2 Homotopy invariants for butterflies

In [1], it is shown that every butterfly induces a span of crossed modules. The related construction is represented in the diagram below:




(9)

where the crossed module $\kappa \sharp \iota$ is the cooperator (see [7]) of the two maps $\kappa$ and $\iota$, which exists since $H$ and $G$ commute in $E$ (see [1] for details), and the morphism $(p_1, \delta)$ is a weak equivalence. The universal property of the bicategory of fractions (see [39]) allows us to extend the definition of $\pi_0$ and $\pi_1$ to butterflies:

**Definition 4.4.** *Let $\mathcal{C}$ be a semi-abelian category satisfying* (SH). *For the butterfly $\widehat{E}\colon \mathbb{H} \looparrowright \mathbb{G}$ in diagram (9) $\pi_{0,1}$ is given by:*

$$\pi_0(\widehat{E}) = \pi_0(p_2, \gamma) \cdot (\pi_0(p_1, \delta))^{-1}\,, \qquad \pi_1(\widehat{E}) = \pi_1(p_2, \gamma) \cdot (\pi_1(p_1, \delta))^{-1}\,.$$

# 5 Crossed extensions

As observed in Section 3.4, every morphism of crossed modules can be factored through a $\pi_0$-cartesian arrow. Indeed, given a crossed module $\mathbb{G} = (\partial_G, \xi_G)$ and a morphism $\phi\colon B \to \pi_0(\mathbb{G})$, one can perform the same construction as in the right hand side of diagram (4), replacing $\pi_0(\underline{f})$ with $\phi$, in order to produce a $\pi_0$-cartesian lifting $\partial'$ of $\phi$ at $\mathbb{G}$. This shows that $\pi_0$ is a fibration only up to isomorphism, since *a priori* $\pi_0(\partial')$ is only isomorphic to and not equal to $B$.

In this section, we show how it is possible to replace $\pi_0$ with a true fibration and, afterwards, how to get a fibration in groupoids out of this, i.e. a fibration whose fibers are groupoids. This procedure will allow us to apply the general framework of Section 2 to the problem of classification of extensions, which is treated in Section 6.3.

First of all, we define the category $\mathbf{XExt}(\mathcal{C})$, whose objects are internal *crossed extensions* in $\mathcal{C}$, i.e. sequences of morphisms in $\mathcal{C}$ of the form:

$$A \xrightarrow{k} G \xrightarrow{\partial_G} G_0 \xrightarrow{q} B$$

where $(\partial_G, \xi_G)$ is a crossed module, $k$ is a kernel of $\partial_G$ and $q$ is a cokernel of $\partial_G$. We use the notation $(\partial_G, \xi_G, A, B)$ or $(\mathbb{G}, A, B)$ to indicate the crossed extension above. A morphism between two objects in $\mathbf{XExt}(\mathcal{C})$ is just a morphism of the underlying crossed modules, together with the induced arrows on the chosen kernels and cokernels. As for crossed modules, we denote morphisms of crossed extensions with underlined letters. A forgetful functor $U\colon \mathbf{XExt}(\mathcal{C}) \to \mathbf{XMod}(\mathcal{C})$ is defined in the obvious way and it is an equivalence of categories. We will freely use the terminology of crossed modules also for crossed extensions, so, for example, a weak equivalence in $\mathbf{XExt}(\mathcal{C})$ is a morphism whose underlying arrows on $A's$ and $B's$ are isomorphisms.

We can define a functor $\Pi_0\colon \mathbf{XExt}(\mathcal{C}) \to \mathcal{C}$ acting on objects as follows (and in the obvious way on arrows):

$$\Pi_0(\mathbb{G}, A, B) = B.$$

It is easy to see that $\Pi_0$ is a fibration. A cartesian lifting at $(\mathbb{G}, A, B)$ of a morphism $\phi\colon B' \to B$ is the following (compare with the right hand side of diagram (4)):

$$\begin{array}{ccccccc} A & \xrightarrow{k} & G & \xrightarrow{\langle 0,\partial_G\rangle} & B' \times_B G_0 & \longrightarrow & B' \\ {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle d_\phi} & & \downarrow{\scriptstyle \phi} \\ A & \xrightarrow[k]{} & G & \xrightarrow[\partial_G]{} & G_0 & \xrightarrow[q]{} & B \end{array} \tag{10}$$

Of course, $\pi_0 \cdot U \cong \Pi_0$. Let us notice that the choice of switching from $\mathbf{XMod}(\mathcal{C})$ (and $\pi_0$) to the equivalent category $\mathbf{XExt}(\mathcal{C})$ (and the equivalent functor $\Pi_0$) is not only convenient, since $\Pi_0$ is a true fibration, but also necessary in order to a cohomology classification of extensions, for which purpose we need to fix the coefficients ($A$ and $B$) once and for all.

However, in order to apply the general framework of Section 2, we need a further step. Let $\mathbf{BExt}(\mathcal{C})$ be the category whose objects are crossed extensions in $\mathcal{C}$ and whose arrows are equivalence classes of butterflies between the underlying crossed modules, together with the induced morphisms on the chosen kernels and cokernels, defined like in Definition 4.4 (one can show that these are well defined on equivalence classes of butterflies).

The homomorphism of bicategories (8) sends every morphism of crossed modules to a butterfly, and this assignment preserves associativity up to 2-cells. Hence, a functor is defined:

$$F\colon \mathbf{XExt}(\mathcal{C}) \to \mathbf{BExt}(\mathcal{C})$$

which is the identity on objects and takes any morphism $\underline{f}$ of crossed extensions to the class $[\mathcal{B}(U\underline{f})]$ of butterflies, together with the induced morphisms on the chosen kernels and cokernels (one can show that these are the same as in the original morphism of crossed extensions).

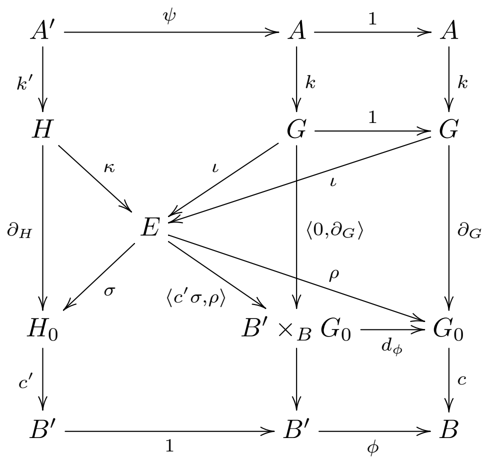
Now, given a butterfly $\widehat{E}$ representing a morphism $[\widehat{E}]$ in $\mathbf{BExt}(\mathcal{C})$, we can associate with it a span $(\underline{s}, \underline{f})$ in $\mathbf{XExt}(\mathcal{C})$, obtained extending the construction of Section 4.2 to crossed extensions. Moreover, since $\underline{s}$ is a weak equivalence, $\mathcal{B}(U\underline{s})$ has a quasi-inverse and, as proved in Theorem 5.6 in [1], $\widehat{E} \cdot \mathcal{B}(U\underline{s}) \cong \mathcal{B}(U\underline{f})$. This implies that $F(\underline{s})$ is invertible and $F(\underline{f}) \cdot F(\underline{s})^{-1} = [\widehat{E}]$.

**Theorem 5.1.** *There exists a unique functor $\widehat{\Pi}_0\colon \mathbf{BExt}(\mathcal{C}) \to \mathcal{C}$ making the following triangle a morphism of fibrations over $\mathcal{C}$:*

$$\begin{array}{ccc} \mathbf{XExt}(\mathcal{C}) & \xrightarrow{F} & \mathbf{BExt}(\mathcal{C}) \\ & {\scriptstyle \Pi_0}\searrow \quad \swarrow{\scriptstyle \widehat{\Pi}_0} & \\ & \mathcal{C} & \end{array}$$

*Proof.* Given a morphism in $\mathbf{BExt}(\mathcal{C})$ represented by a butterfly $\widehat{E}$, with $(\underline{s}, \underline{f})$ a corresponding span in $\mathbf{XExt}(\mathcal{C})$, we are forced to define $\widehat{\Pi}_0([\widehat{E}]) = \Pi_0(\underline{f}) \cdot \Pi_0(\underline{s})^{-1}$.

Now, let $\phi\colon B' \to B$ be a morphism in $\mathcal{C}$, $(\mathbb{G}, A, B)$ an object in $\mathbf{XExt}(\mathcal{C})$ and let us call $\underline{d}_\phi$ the cartesian lifting of $\phi$ at $(\mathbb{G}, A, B)$ described above. Suppose $[\widehat{E}]\colon (\mathbb{H}, A', B') \to (\mathbb{G}, A, B)$ is such that $\widehat{\Pi}_0([\widehat{E}]) = \phi$. If $(\underline{s}, \underline{f})$ is a span corresponding to $\widehat{E}$, as above, then $\Pi_0(\underline{f}) = \phi \cdot \Pi_0(\underline{s})$. Since $\underline{d}_\phi$ is cartesian, there exists a unique $\underline{h}$ such that $\underline{f} = \underline{d}_\phi \cdot \underline{h}$ and $\Pi_0(\underline{h}) = \Pi_0(\underline{s})$. Moreover, it is easy to see that the span $(\underline{s}, \underline{h})$ is also associated with a butterfly. It suffices to consider the following diagram:




In order to prove that the 4-tuple $(\iota, \sigma, \kappa, \langle c'\sigma, \rho\rangle)$ still forms a butterfly, we only have to show that $\langle c'\sigma, \rho\rangle \cdot \kappa = 0$. But this is true by composition with the jointly monomorphic pair of projections of the pullback $B' \times_B G_0$. We denote this new butterfly as $\widehat{E'}$. Completing the diagram with the corresponding span of crossed modules and extending it suitably to crossed extensions, we get precisely the span $(\underline{s}, \underline{h})$.

We are now ready to prove that $F(\underline{d}_\phi)$ is a cartesian lifting of $\phi$ at $\mathbb{G}$ with respect to $\widehat{\Pi}_0$. Indeed, by functoriality, $F(\underline{f}) = F(\underline{d}_\phi) \cdot F(\underline{h})$ and we have the following chain of equalities in $\mathbf{BExt}(\mathcal{C})$:

$$F(\underline{d}_\phi) \cdot [\widehat{E'}] = F(\underline{d}_\phi) \cdot F(\underline{h}) \cdot F(\underline{s})^{-1} = F(\underline{f}) \cdot F(\underline{s})^{-1} = [\widehat{E}].$$

This proves that any morphism $[\widehat{E}]$ in $\mathbf{BExt}(\mathcal{C})$ such that $\widehat{\Pi}_0([\widehat{E}]) = \phi$ factors through $F(\underline{d}_\phi)$ in a unique way by a morphism $[\widehat{E'}]$ with $\widehat{\Pi}_0([\widehat{E'}]) = 1$.

In conclusion, we have just shown that $\widehat{\Pi}_0$ is a fibration and $F$ preserves cartesian morphisms. □

In the same way we defined $\Pi_0$ to extend $\pi_0$, we can also define a functor $\Pi_1\colon \mathbf{XExt}(\mathcal{C}) \to \mathcal{C}$ acting on objects as follows (and in the obvious way on arrows):

$$\Pi_1(\mathbb{G}, A, B) = A.$$

Clearly, $\pi_1 \cdot U \cong \Pi_1$, and this functor also extends to butterflies in the same way as $\widehat{\Pi}_0$:

$$\widehat{\Pi}_1 \colon \mathbf{BExt}(\mathcal{C}) \to \mathcal{C}, \qquad \widehat{\Pi}_1([\widehat{E}]) = \Pi_1(\underline{f}) \cdot \Pi_1(\underline{s})^{-1},$$

where, as above, $(\underline{s}, \underline{f})$ is a span in $\mathbf{XExt}(\mathcal{C})$ associated with $\widehat{E}$.

Since it will be necessary later, we focus now on the kernel of $\widehat{\Pi}_1$, i.e. the full subcategory of $\mathbf{BExt}(\mathcal{C})$ given by those morphisms which are mapped to identities by $\widehat{\Pi}_1$. We denote this subcategory by $I \colon \mathbf{Ker}(\widehat{\Pi}_1) \to \mathbf{BExt}(\mathcal{C})$.

**Proposition 5.2.** *The functor $\widehat{\Pi}_0 \cdot I \colon \mathbf{Ker}(\widehat{\Pi}_1) \to \mathcal{C}$ is a fibration in groupoids.*

*Proof.* The construction of diagram (10), together with the observation that $\widehat{\Pi}_0 \cdot F = \Pi_0$, explain that cartesian liftings with respect to $\widehat{\Pi}_0$ can be chosen to induce identity on $\widehat{\Pi}_1$, and this proves that $\widehat{\Pi}_0 \cdot I$ is also a fibration. Moreover, the fibers are formed by equivalence classes of butterflies inducing identities on $\widehat{\Pi}_0$ and $\widehat{\Pi}_1$, i.e. invertible arrows in $\mathbf{BExt}(\mathcal{C})$. □

# 6 Extensions

A connection between the cohomology classification of group extensions and the notion of monoidal functor has been already recalled in the introduction (see [41] for a detailed account). More specifically, as we are going to show, it is possible to embed group extensions into some specific hom-categories of **2Gp**. This can be efficiently described in terms of crossed modules, and furthermore the same scheme applies in the internal setting.

## 6.1 Extensions as butterflies

For groups $K$ and $Y$, let $\mathrm{AUT}(K)$ be the groupoid associated with the canonical crossed module of inner automorphisms

$$\mathcal{I}_K \colon K \to \mathrm{Aut}(K)\,,$$

sending each element $k$ of $K$ to the automorphism $(x \mapsto k^{-1}xk)$, and let $D(Y)$ be the discrete groupoid associated with the canonical trivial crossed module

$$\Delta_Y \colon 0 \to Y\,.$$

There is an isomorphism of categories

$$\mathrm{EXT}(Y, K) \cong \mathbf{2Gp}(D(Y), \mathrm{AUT}(K))$$

between the groupoid of extensions of $Y$ by $K$ and the hom-groupoid of monoidal functors from $D(Y)$ to $\mathrm{AUT}(K)$.

In the language of butterflies, this isomorphism sends the short exact sequence

$$K \xrightarrow{k} X \xrightarrow{f} Y$$

to the butterfly $\widehat{X}\colon \Delta_Y \nrightarrow \mathcal{I}_K$:

$$\begin{array}{ccccc} 0 & & & & K \\ \big\downarrow & \searrow & & \swarrow^{k} & \big\downarrow \mathcal{I}_K \\ & & X & & \\ \big\downarrow & \swarrow_{f} & & \searrow^{\chi} & \big\downarrow \\ Y & & & & \mathrm{Aut}(K) \end{array} \tag{11}$$

where $\chi$ is the group homomorphism corresponding to the conjugation action of $X$ on its normal subgroup $K$. One can always complete the diagram above to a butterfly of crossed extensions, getting the trivial map $0 \to Z(K)$ on kernels, and, on cokernels, the so called *abstract kernel* of the extension:

$$\phi\colon Y \to \mathrm{Out}(K)\,,$$

where $\mathrm{Out}(K) = \mathrm{Aut}(K)/\mathrm{Inn}(K)$.

**Remark 6.1.** From this point of view, the *cohomology* classification of the extensions inducing a given abstract kernel $\phi$ becomes an instance of the *homotopy* classification of weak maps between groupoids in groups (see for example [3]). We recall that the last has been studied (and exhaustively described) by Vitale in [41].

The previous discussion carries on almost verbatim in all semi-abelian categories with representable object actions (see [9] and also [8], where they are called *action representative*), i.e. those semi-abelian categories where, for any $K$, the functor $\mathbf{Act}(-, K)$ (see Section 3.1) is represented by a suitable object $[K]$, called the *actor*, as for $\mathrm{Aut}(K)$ in the case of groups.

We denote by

$$\mathcal{I}_K\colon K \to [K]$$

the crossed module associated with the actor determined by $K$.

For any $Y$ and $K$ in an action representative semi-abelian category $\mathcal{C}$, we identify the groupoid:

$$\mathrm{EXT}(Y, K) = \mathbf{Bfly}(\mathcal{C})(\Delta_Y, \mathcal{I}_K)\,.$$

We use lowercase letters for its classifying set $\mathrm{Ext}(Y, K)$. Finally, for a given abstract kernel $\phi$, we use the notation $\mathrm{OPEXT}(Y, K, \phi)$ and $\mathrm{OpExt}(Y, K, \phi)$ for the groupoid, and the set of equivalence classes, of extensions inducing $\phi\colon Y \to \mathrm{Out}(K)$.

In order to emphasize the connection with the classical case of extensions with abelian kernel, we speak of extensions of $Y$ with coefficients in the crossed module $\mathcal{I}_K$, inducing the abstract kernel $\phi$, so that abuses of notation such as $\mathrm{OpExt}(Y, \mathcal{I}_K, \phi)$ or $\mathrm{OpExt}(Y, \mathbb{K}, \phi)$ will be allowed.

If the semi-abelian category $\mathcal{C}$ is not action representative, but it is still action accessible (see [17]), we do not have, in general, $\mathcal{I}_K$, for every $K$. Nevertheless, the theory can be developed very closely to the action representative case (see [18, 22]) by considering the crossed module associated with the canonical faithful groupoid determined by the extension.

Actually, a cohomological classification of weak maps can be performed also when the base category is not even action accessible. On the side of extensions, generalizing the approach described above, we will be able to classify extensions with coefficients in an arbitrary crossed module, inducing a given abstract kernel. Before developing such a classification, we shall revisit the classical Baer sums.

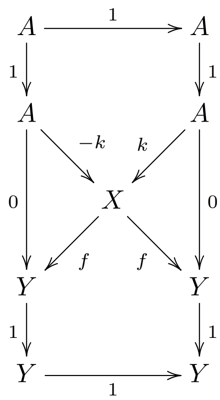
## 6.2 Baer sums as butterfly compositions

Let us consider a short exact sequence of groups with abelian kernel:

$$A \xrightarrow{k} X \xrightarrow{f} Y$$

It is well known that $A$ is endowed with a $Y$-module structure $\phi \colon Y \flat A \to A$. The groupoid of extensions of $Y$ by $A$ inducing the same structure $\phi$ is denoted by $\mathrm{OPEXT}(Y, A, \phi)$. It is a classical result (see [31], for instance) that its classifying set $\mathrm{OpExt}(Y, A, \phi)$ is endowed with an abelian group structure, the group operation given by the Baer sum. Indeed, this abelian group structure comes from a monoidal structure $\oplus$ that makes $\mathrm{OPEXT}(Y, A, \phi)$ into a symmetric categorical group.

Baer sums can be easily described in terms of butterflies, since butterfly composition is actually a generalization of the Baer sum construction. Indeed, an extension $(k, f)$ determines canonically, together with the action $\phi$, a butterfly of crossed extensions:




(12)

It is a simple, but interesting, exercise to show that, if we consider another extension $(k', f')$ in $\mathrm{OPEXT}(Y, A, \phi)$, their composite $\widehat{X}' \cdot \widehat{X}$ as butterflies corresponds to their *Baer sum* $(k, f) \oplus (k', f')$. This way, we get a group homomorphism

$$\mathrm{OpExt}(Y, A, \phi) \to \mathbf{BExt}(\mathbf{Gp})((0, \phi, Y, A), (0, \phi, Y, A)).$$

The discussion above can be internalized. Let $\mathcal{C}$ be a semi-abelian category satisfying (SH). As for the case of groups, we can still embed extensions with abelian kernel as butterflies described by diagram (12). In this a case, we have $\widehat{\Pi}_0(\widehat{X}) = 1_Y$ and $\widehat{\Pi}_1(\widehat{X}) = 1_A$, so that we shall denote by $\mathbf{BExt}_1(\mathcal{C})((0,\phi,Y,A),(0,\phi,Y,A))$ the group of (equivalence classes of) such butterflies. This group bijectively corresponds to the group $\mathrm{OpExt}(Y,A,\phi)$ of extensions with abelian kernel inducing $\phi\colon Y\flat A \to A$. The proof is a consequence of the following lemma.

**Lemma 6.2.** *Let $A$ be a $Y$-module, with action $\phi\colon Y\flat A \to A$, and*

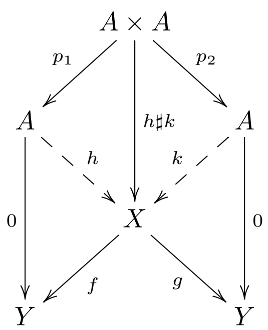
$$\widehat{X}\colon (0,\phi,Y,A) \looparrowright (0,\phi,Y,A)$$

*a butterfly such that*

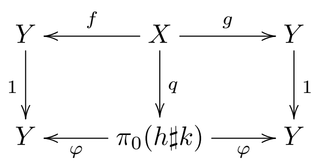
$$\widehat{\Pi}_0(\widehat{X}) = 1_Y\,, \qquad \widehat{\Pi}_1(\widehat{X}) = 1_A\,.$$

*Then $\widehat{X}$ is of the form of diagram (12).*

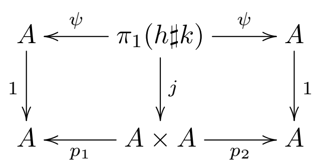
*Proof.* Let $\widehat{X}$ be as in the hypothesis, and consider its corresponding span of crossed module morphisms:




Then, on cokernels, since $\widehat{\Pi}_0(\widehat{X}) = \Pi_0(p_2,g)\cdot(\Pi_0(p_1,f))^{-1} = 1_Y$, we get a corestriction map $\varphi$ such that the following diagram commutes:




hence $f = \varphi\cdot q = g$. Similarly, on kernels, we get a restriction map $\psi$ such that the following diagram commutes:



Then $p_1 \cdot j \cdot \psi^{-1} = p_2 \cdot j \cdot \psi^{-1} = 1_A$, i.e. $j \cdot \psi^{-1} = \langle 1, 1 \rangle$, and as a consequence $h + k = (h \sharp k) \cdot \langle 1, 1 \rangle = (h \sharp k) \cdot j \cdot \psi^{-1} = 0$, that is $h = -k$. □

In the following, we will denote

$$H^2(Y, A, \phi) = \mathbf{BExt}_1(\mathcal{C})((0, \phi, Y, A), (0, \phi, Y, A)).$$

Notice that this definition is consistent with the crossed module version of Bourn's groupoid cohomology of [15].

## 6.3 Cohomology classification of extensions with coefficients in a crossed module

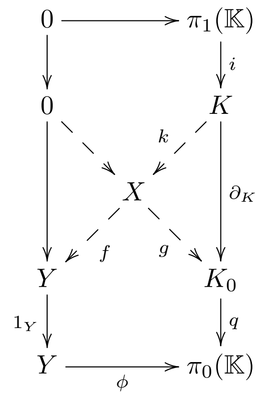
In a semi-abelian category $\mathcal{C}$ satisfying (SH), we consider the following problem: given a crossed module

$$\mathbb{K} = (\ K_0 \flat K \xrightarrow{\xi_K} K \xrightarrow{\partial_K} K_0\ ),$$

and a morphism

$$Y \xrightarrow{\phi} \pi_0(\mathbb{K})\ ,$$

determine the set $\mathrm{OpExt}(Y, \mathbb{K}, \phi)$ of all the extensions of $Y$ by $\mathbb{K}$ inducing $\phi$. In the spirit of Section 6.1, this amounts to classifying all the butterflies that fits into the diagram:



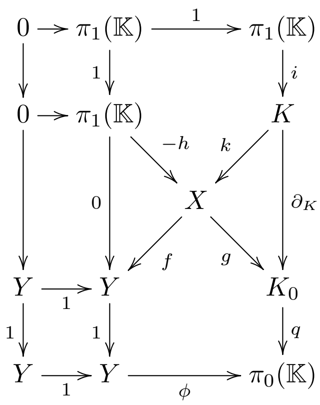
(13)

The answer is a consequence of the following result.

**Theorem 6.3.** *In a semi-abelian category $\mathcal{C}$ satisfying* (SH)*, we consider a crossed module $\mathbb{K} = (\partial_K, \xi_K)$, together with an arrow $\phi \colon Y \to \pi_0(\mathbb{K})$. Then, either* $\mathrm{OpExt}(Y, \mathbb{K}, \phi)$ *is empty, or it is a simply transitive $H^2\big(Y, \pi_1(\mathbb{K}), \overline{\phi}\big)$-set, where*

- *$\overline{\phi} = \phi^*(\overline{\xi}_K)$ is the pullback along $\phi$ of the action of $\pi_0(\mathbb{K})$ on $\pi_1(\mathbb{K})$ induced by $\xi_K$ (see Lemma 3.2).*

*Proof.* Suppose that the set $\mathrm{OpExt}(Y, \mathbb{K}, \phi)$ is not empty, and let $(k, f, g)$ be an extension of $Y$ by $\mathbb{K}$ inducing $\phi$, as in diagram (13). Then, denoting $h = k \cdot i$, we get a factorization of the butterfly (13):




(14)

(observe that as a kernel of $h\sharp k$ we can choose $\langle 1, i\rangle \colon \pi_1(\mathbb{K}) \to \pi_1(\mathbb{K}) \times K$). This factorization provides a canonical way to associate with any extension $(k, f, g)$ a butterfly between crossed extensions inducing an identity on kernels.

Let now $\mathcal{E}$ be the category $\mathbf{Ker}(\widehat{\Pi}_1)$ of (equivalence classes of) butterflies between crossed extensions inducing identity on kernels, introduced in Section 5, and $\Pi$ the functor $\widehat{\Pi}_0 \cdot I$. Proposition 5.2 says precisely that $\Pi \colon \mathcal{E} \to \mathcal{C}$ is a fibration in groupoids.

Moreover, if we take $x$ and $y$ to be the domain and codomain of the butterfly on the right hand side of diagram (14), i.e. the crossed extensions $(0, \overline{\phi}, Y, \pi_1(\mathbb{K}))$ and $(\partial_K, \xi_K, \pi_0(\mathbb{K}), \pi_1(\mathbb{K}))$ respectively, then the factorization above yields a bijection:

$$\mathrm{OpExt}(Y, \mathbb{K}, \phi) \cong \mathcal{E}_\phi(x, y).$$

We are now ready to apply Proposition 2.7 to our data $\Pi$, $\phi$, $x$ and $y$. Thanks to Lemma 6.2, the group $G$, in this case, is just $H^2\big(Y, \pi_1(\mathbb{K}), \overline{\phi}\big)$. The thesis follows from point $(ii)$ of Proposition 2.7. □

We can reformulate the theorem as follows.

**Corollary 6.4.** *With the same hypotheses and notation of Theorem 6.3, if we denote by $\Phi$ the set of all $\psi \colon Y \to \pi_0(\mathbb{K})$ such that $\overline{\psi} = \overline{\phi}$, then either the set*

$$S = \coprod_{\psi \in \Phi} \mathrm{OpExt}(Y, \mathbb{K}, \psi)$$

*is empty, or it is an $H^2\big(Y, \pi_1(\mathbb{K}), \overline{\phi}\big)$-torsor over its image*

$$\pi_0(S) \subset \mathcal{C}(Y, \pi_0(\mathbb{K}))$$

*Proof.* Point $(i)$ of Proposition 2.7. □

Finally, we recover an intrinsic version of the classical Schreier-Mac Lane theorem on the classification of non-abelian extensions. The same result was proved by Bourn in [13] in the (larger) context of exact action representative categories.

**Corollary 6.5.** *Let $\mathcal{C}$ be an action representative semi-abelian category. Then, following the notation of Section 6.1, we consider two objects $Y$ and $K$, together with an arrow $\phi\colon Y \to \pi_0(\mathcal{I}_K)$. Then, either* $\mathrm{OpExt}(Y, K, \phi)$ *is empty, or it is a simply transitive $H^2(Y, Z(K), \overline{\phi})$-set, where*

- *$Z(K) = \pi_1(\mathcal{I}_K)$ (the center of $K$),*
- *$\overline{\phi}$ is the pullback along $\phi$ of the (canonical) action of $\pi_0(\mathcal{I}_K)$ on $Z(K)$.*

*Proof.* In this context, one can apply Theorem 6.3 to the case where $\mathbb{K} = \mathcal{I}_K$. Moreover, the set $\mathrm{OpExt}(Y, K, \phi)$ of extensions inducing the abstract kernel $\phi$ is in bijection with $\mathrm{OpExt}(Y, \mathcal{I}_K, \phi)$, since there is no arbitrary choice for the crossed module $K \to [K]$. □

## 6.4 An explicit description of the action

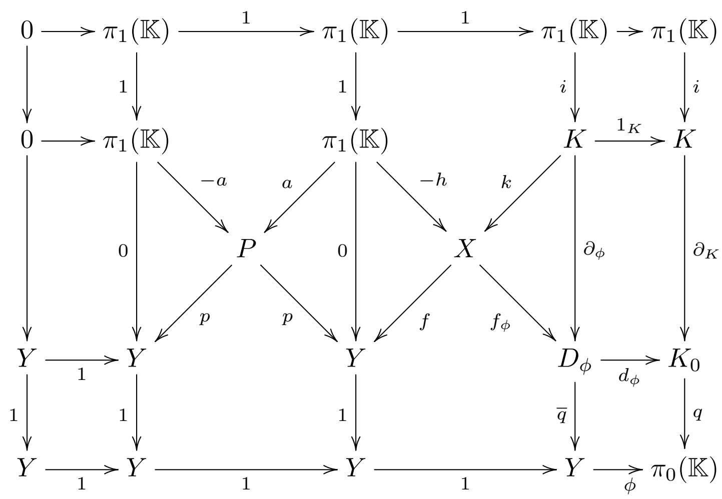
It is now possible to describe the action

$$H^2\left(Y, \pi_1(\mathbb{K}), \overline{\phi}\right) \times \mathrm{OpExt}(Y, \mathbb{K}, \phi) \to \mathrm{OpExt}(Y, \mathbb{K}, \phi)$$

For (a cohomology class determined by) an extension

$$\pi_1(\mathbb{K}) \xrightarrow{a} P \xrightarrow{p} Y$$

in $H^2\left(Y, \pi_1(\mathbb{K}), \overline{\phi}\right)$ it is obtained by the following composition:




(15)

where the crossed module morphism $(1_K, d_\phi)$ is a $\pi_0$-cartesian lifting of $\phi$ and $f_\phi$ is the obvious comparison.

Hence, by Lemma 2.1, the action comes from the free transitive action obtained by composition in $\mathbf{BExt}(\mathcal{C})$:

$$\mathbf{BExt}_1(\mathcal{C})(\Phi,\Phi)\times \mathbf{BExt}_1(\mathcal{C})(\Phi,\Xi) \longrightarrow \mathbf{BExt}_1(\mathcal{C})(\Phi,\Xi)$$

where $\Phi = (0, \phi, Y, \pi_1(\mathbb{K}))$ and $\Xi = (\partial_\phi, \xi_\phi, Y, \pi_1(\mathbb{K}))$. The details are left to the reader.

## 6.5 Obstruction

Theorem 6.3 describes the set $\mathrm{OpExt}(Y, \mathbb{K}, \phi)$ when it is not empty, so it is natural to ask how to distinguish the empty case from the non-empty one. The answer to this question can still be expressed in terms of cohomology classes. In the same way as in [13] and [22], we can apply Bourn's $n$-groupoid cohomology theory to our situation (we refer the interested reader to [11] or [15] for a more detailed account). Accordingly, we can interpret every crossed module $K \to K_0$, or more precisely its corresponding groupoid $\mathbb{K}$, as an element of a certain cohomology group, namely $H^2_{\mathcal{C}\downarrow\pi_0(\mathbb{K})}(\mathbb{K})$. In the following we will denote this group $H^3(\pi_0(\mathbb{K}), \pi_1(\mathbb{K}), \overline{\xi})$, where $\overline{\xi}$ is the action of $\pi_0(\mathbb{K})$ on $\pi_1(\mathbb{K})$ determined by the crossed module $\mathbb{K}$. This notation is more consistent with the classical cohomology theories. Indeed, one can show that, when $\mathcal{C}$ is the category of groups, the object above coincides with the corresponding classical third cohomology group. The same happens, for example, with Hochschild cohomology groups of associative algebras and with Loday-Pirashvili cohomology groups of Leibniz algebras over a field (see [27] and [30] respectively).

From this point of view, $H^3(Y, \pi_1(\mathbb{K}), \overline{\phi})$ is defined as the set (or, more precisely, the abelian group) of connected components of $\Pi^{-1}_{0,1}(Y, \pi_1(\mathbb{K}), \overline{\phi})$, i.e. the full subcategory of $\mathbf{XExt}(\mathcal{C})$ given by those objects whose global direction is the module $(0\colon \pi_1(\mathbb{K}) \to Y, \overline{\phi})$ (a global direction on crossed extensions can be defined likewise in Lemma 3.2) and those morphisms inducing identities on kernels and cokernels. This way, a class $[\partial]$ in $H^3(Y, \pi_1(\mathbb{K}), \overline{\phi})$ is zero if and only if $(\partial, \xi)$ is connected with $(0, \overline{\phi})$. Then we get the following result.

**Theorem 6.6.** *In a semi-abelian category $\mathcal{C}$ satisfying* (SH)*, we consider a crossed module $\mathbb{K} = (\partial_K, \xi_K)$, together with an arrow $\phi\colon Y \to \pi_0(\mathbb{K})$. Then, with the same notation as in Theorem 6.3, the set* $\mathrm{OpExt}(Y, \mathbb{K}, \phi)$ *is non-empty if and only if $[\partial_\phi] = 0$.*

*Proof.* Suppose that an extension $(k, f, g)$ of $Y$ by $\mathbb{K}$ exists, such that $\phi$ is its abstract kernel. As in the description of the action given above, we can construct an equivalence between $(0, \overline{\phi})$ and $(\partial_\phi, \xi_\phi)$ (the butterfly $(-h, k, f, f_\phi)$ in diagram (15)), hence the cohomology class of $\partial_\phi$ is trivial.

Now, following the proof of Theorem 3.4 in [13] and translating the argument therein in terms of crossed modules, thanks to Theorem 12 in [12], we get that, whenever the cohomology class of $\partial_\phi$ is trivial, there exists a morphism $(u, u_0)$ of crossed modules

$$\begin{array}{ccc} H & \xrightarrow{u} & K \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \partial_\phi} \\ H_0 & \xrightarrow[u_0]{} & D_\phi \end{array}$$

where $i$ is a normal monomorphism with $\mathrm{Coker}(i) \cong Y$. Computing the (final, discrete fibration) factorization of $(u, u_0)$, we obtain a normal monomorphism $k$, again with $\mathrm{Coker}(k) \cong Y$ (i.e. the push forward of $i$ along $u$), and the discrete fibration on the right hand side of the following commutative diagram

$$\begin{array}{ccccc} H & \xrightarrow{u} & K & \xrightarrow{1_K} & K \\ {\scriptstyle i}\downarrow & & {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \partial_\phi} \\ H_0 & \xrightarrow{e} & X & \xrightarrow{m} & D_\phi \end{array} \qquad u_0 = m \circ e$$

Taking $f : X \to Y$ a cokernel of $k$ and $g = d_\phi \cdot m$, the triple $(k, f, g)$ provides an extension of $Y$ by $\mathbb{K}$ inducing $\phi$ as abstract kernel. □

### 6.6 A final remark

In his *Historical Note* [32] concerning the long-lasting quest for group-theoretical interpretations for the cohomology groups $H^n(G, A)$, Mac Lane acknowledges that, even before the exact theorem was explicitly stated by Gerstenhaber in [26] (see also [25]), "Eilenberg, Mac Lane and Whitehead all knew that the elements of $H^3(G, A)$ were closely connected with Whitehead's notion of "crossed modules" ". In fact, the exposition of the subject given in his fundamental book *Homology* [31], presents the classical interpretations of $H^1$ and $H^2$ in terms of crossed homomorphisms and factor sets, and the interpretation of $H^3$, classical nowadays, in terms of obstructions to the construction of group extensions with non-abelian kernel. Nonetheless, in [31, IV.8], after the explicit cocycle description of the action of $H^2$ on OpExt, he concludes the section asserting that such an action "may also be defined in invariant terms, without using factor sets". What follows in his text is, *mutatis mutandis*, precisely the construction underlying the butterfly composition displayed in diagram (15) above, specialized in the case of groups.

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