Butterflies in a semi-abelian context

In memory of Aurelio Carboni

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Abstract

It is known that monoidal functors between internal groupoids in the category *Grp* of groups constitute the bicategory of fractions of the 2-category *Grpd*(*Grp*) of internal groupoids, internal functors and internal natural transformations in *Grp*, with respect to weak equivalences (that is, internal functors which are internally fully faithful and essentially surjective on objects). Monoidal functors can be equivalently described by a kind of weak morphisms introduced by B. Noohi under the name of butterflies. In order to internalize monoidal functors in a wide context, we introduce the notion of internal butterflies between internal crossed modules in a semi-abelian category $\mathcal{C}$, and we show that they are morphisms of a bicategory $\mathcal{B}(\mathcal{C})$. Our main result states that, when in $\mathcal{C}$ the notions of Huq commutator and Smith commutator coincide, then the bicategory $\mathcal{B}(\mathcal{C})$ of internal butterflies is the bicategory of fractions of *Grpd*($\mathcal{C}$) with respect to weak equivalences.

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1. Introduction

A groupoid in the category of groups is a special case of strict monoidal category, tensor product being provided by the group structure on objects and arrows. Therefore, beyond internal functors, as arrows between groupoids in groups we can consider monoidal functors, that is, functors between the underlying categories

$$F\colon \mathbb{H} \to \mathbb{G},$$

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equipped with a natural and coherent family of isomorphisms

$$F^{x,y}\colon F_0(x)+F_0(y)\to F_0(x+y) \qquad x,y\in H_0\,.$$

Both notions of monoidal functor and internal functor are relevant as morphisms of groupoids in groups (just to cite an example, as special case of monoidal functors we get group extensions, whereas in the same case internal functors give split extensions, see Section 7), so the question of expressing monoidal functors in an internal way arises.

Three progresses have been recently accomplished in this direction. In [40] (see also [41] and [2]) B. Noohi has proved that the bicategory having groupoids in groups as objects and monoidal functors as 1-cells can be equivalently described using crossed modules of groups as objects and what he calls butterflies as arrows. Moreover, in a paper with E. Aldrovandi [2], the theory is pushed forward in order to include the more general situation where groups are replaced by internal groups in a Grothendieck topos. Noohi's butterflies (of [40]) have been studied in the case of Lie algebras on a field in [1], and in [42], where it is proved that butterflies between differential crossed modules (i.e. crossed modules of Lie algebras) represent homomorphisms of strict Lie 2-algebras.

On the other hand, in [48] it has been proved that the bicategory of groupoids in groups and monoidal functors is the bicategory of fractions of the bicategory of groupoids in groups and internal functors with respect to weak equivalences. Once again, the same result holds replacing groups with Lie algebras and monoidal functors with homomorphisms of strict Lie 2-algebras. In [24], M. Dupont has proved that butterflies provide the bicategory of fractions of internal functors with respect to weak equivalences when working internally to any abelian category.

The aim of this paper is to unify the results in [40], [1], [42], [24] and [48]. We introduce and study the bicategory $\mathcal{B}(\mathcal{C})$ of crossed modules and butterflies in a semi-abelian category $\mathcal{C}$. The main result is Theorem 5.6, where we prove that $\mathcal{B}(\mathcal{C})$ is the bicategory of fractions of the bicategory of groupoids and internal functors with respect to weak equivalences. This result gives a general answer to the specific problem recalled above: butterflies are a representation of weak internal functors. They generalize at once monoidal functors in *Grpd*(*Grp*) and homomorphisms in *Grpd*(k*Lie*), and they work for other 2-dimensional algebraic settings, as for groupoids of Leibniz algebras, associative algebras, rings etc. Actually, butterflies come out of the notion of internal profunctor, by means of a process of normalization, as described in [38], where the non-pointed case is examined in details. In fact, in [38] the whole story has been told in terms of fractors (see Section 3.2): fractors are a special kind of internal profunctors corresponding to butterflies (when $\mathcal{C}$ is semi-abelian) and providing the bicategory of fractions of groupoids with respect to weak equivalences in the more general context of Barr exact categories. The case of internal categories is treated by D. Roberts in [45] using internal anafunctors.

A few lines on the chosen context. We work internally to a semi-abelian category in which the notions of Huq commutator [30] and Smith commutator [46] coincide. This allows us, among other things, to use a simplified version of internal crossed modules without loosing the equivalence with internal groupoids (see [39]). The categories of groups, Lie algebras, rings and many other algebraic structures not only in *Set*, but in any Grothendieck topos satisfy this condition (see Section 9.3), so that our context includes also that of [2].

Finally, let us give a glance to possible developments of the present work. Quite a lot of higher dimensional group theory has been developed starting from the pioneer works of P. Deligne [23], A. Fröhlich and C.T.C. Wall [27] on Picard categories (also called 2-groups or categorical groups), taking monoidal functors as morphisms (see for example [47], [2], [25] and the references therein). On the other hand, group theory has been the paradigmatic example to develop in recent years semi-abelian categorical algebra (see Section 9 and the references therein). The fact of disposing of a normalized internal notion of monoidal functor should make it possible to join these two generalizations of group theory and to develop a kind of higher dimensional semi-abelian categorical algebra which could cover as special cases most of the known results on (strict) categorical groups and (strict) Lie 2-algebras.

The layout of this paper is as follows: in Section 2 we recall the equivalence between internal groupoids and internal crossed modules, a result due to G. Janelidze (see [31]) and which holds in any semi-abelian category; in Sections 3 and 4 we study the bicategory $\mathcal{B}(\mathcal{C})$ of butterflies in a semi-abelian category $\mathcal{C}$ with "Huq = Smith"; in Section 5 we prove that $\mathcal{B}(\mathcal{C})$ is the bicategory of fractions of internal functors with respect to weak equivalences; in Section 6 we examine the two leading examples of groups and Lie algebras; Section 7 is a short section devoted to the classification of extensions which follows from Section 5; in Section 8 we specialize the main result of Section 5 to the case where $\mathcal{C}$ is a free exact category; finally, Section 9 is a reminder on protomodular and semi-abelian categories. The reader who is not familiar with semi-abelian categories should have a glance to Section 9 before reading Section 2.

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Terminology: bicategory means bicategory with invertible 2-cells.

Contents

2. Internal groupoids and internal crossed modules

An introduction to internal categories can be found in Chapter 8 of [7]. For basic facts on 2-categories and bicategories, see [5] or Chapter 7 in [7].

2.1. Internal groupoids

Let $\mathcal{C}$ be a category with finite limits. We use the following notation:

1. An (internal) groupoid $\mathbb{G}$ in $\mathcal{C}$ is displayed as

$$G_1 \times_{c,d} G_1 \xrightarrow{m} G_1 \underset{c}{\overset{d}{\underset{\longleftarrow e}{\rightrightarrows}}} G_0 \qquad G_1 \xrightarrow{i} G_1$$

where

$$\begin{tikzcd} G_1 \times_{c,d} G_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & G_1 \arrow[d, "d"] \\ G_1 \arrow[r, "c"'] & G_0 \end{tikzcd}$$

is a pullback.

2. An (internal) functor $P = (p_1, p_0)\colon \mathbb{G} \to \mathbb{H}$ is displayed as

$$\begin{tikzcd} G_1 \arrow[r, "p_1"] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & H_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ G_0 \arrow[r, "p_0"'] & H_0 \end{tikzcd}$$

3. An (internal) natural transformation $\alpha\colon P \Rightarrow Q\colon \mathbb{G} \to \mathbb{H}$ is displayed as

$$\begin{tikzcd} G_1 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "q_1"'] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & H_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ G_0 \arrow[r, shift left, "p_0"] \arrow[r, shift right, "q_0"'] \arrow[ur, "\alpha"] & H_0 \end{tikzcd}$$

Internal groupoids, functors and natural transformations form a 2-category (with invertible 2-cells) denoted by $Grpd(\mathcal{C})$.

When dealing with internal structures, it is sometimes useful to use virtual objects and arrows as if those would be internal to the category of sets. For instance, we could describe the object $G_1 \times_{c,d} G_1$ as the "set" of composable arrows

$$\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$$

Yoneda embedding makes this precise, as explained in [10], Metatheorem 0.2.7.

Definition 2.1. An internal functor $P\colon \mathbb{G} \to \mathbb{H}$ as above is called a *discrete cofibration* when the commutative square $d \cdot p_1 = p_0 \cdot d$ is a pullback. Dually, P is a *discrete fibration* when the square $c \cdot p_1 = p_0 \cdot c$ is a pullback.

Observe that for groupoids the notions of fibration and cofibration are equivalent.

Example 2.2. Consider the diagram

$$\begin{tikzcd} R_c \arrow[r, "\widetilde{d}"] \arrow[d, shift right, "c_1"'] \arrow[d, shift left, "c_2"] & G_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ G_1 \arrow[r, "d"'] & G_0 \end{tikzcd}$$

where (R_c, c_1, c_2) is a kernel pair of c, and $\widetilde{d}$ is the morphism that sends the pair of converging virtual arrows

$$x \xrightarrow{f} y \xleftarrow{g} z$$

in the composition $g^{-1} \cdot f\colon x \to z$. The pair $(\widetilde{d}, d)$ is a discrete fibration of groupoids. A similar argument can be developed for R_d.

The following notion is due to M. Bunge and R. Paré, see [20].

Definition 2.3. Assume that the category $\mathcal{C}$ is regular [4]; an internal functor $P \colon \mathbb{G} \to \mathbb{H}$ is a *weak equivalence* if it is

1. (internally) *full and faithful*, that is, the diagram

$$\begin{tikzcd}
 & & G_1 \arrow[dll, "d"'] \arrow[d, "p_1"] \arrow[drr, "c"] & & \\
G_0 \arrow[dr, "p_0"'] & & H_1 \arrow[dl, "d"] \arrow[dr, "c"'] & & G_0 \arrow[dl, "p_0"] \\
 & H_0 & & H_0 &
\end{tikzcd}$$

   is a limit, and
2. (internally) *essentially surjective on objects*, that is, the composition

$$G_0 \times_{p_0,c} H_1 \xrightarrow{t_2} H_1 \xrightarrow{d} H_0$$

   is a regular epimorphism, where

$$\begin{tikzcd}
G_0 \times_{p_0,c} H_1 \arrow[r, "t_2"] \arrow[d, "t_1"'] & H_1 \arrow[d, "c"] \\
G_0 \arrow[r, "p_0"'] & H_0
\end{tikzcd}$$

   is a pullback.

Observe that $P \colon \mathbb{G} \to \mathbb{H}$ is (internally) full and faithful if and only if, for every groupoid $\mathbb{A}$, the functor

$$P \cdot - \, : \, \mathit{Grpd}(\mathcal{C})(\mathbb{A}, \mathbb{G}) \to \mathit{Grpd}(\mathcal{C})(\mathbb{A}, \mathbb{H})$$

is full and faithful in the usual sense. Moreover, a weak equivalence P is an equivalence if and only if

$$G_0 \times_{p_0,c} H_1 \xrightarrow{t_2} H_1 \xrightarrow{d} H_0$$

is a split epimorphism.

2.2. Internal crossed modules

From now on we assume that $\mathcal{C}$ is a semi-abelian category in which the condition "Huq = Smith" holds (for undefined notions and notation concerning semi-abelian categories, the reader is addressed to Section 9).

An (internal) crossed module $\mathbb{G}$ in $\mathcal{C}$ is given by a morphism $\partial\colon G \to G_0$ and an action $\xi\colon G_0 \flat G \to G$ such that the diagram

$$\begin{tikzcd}
G \flat G \arrow[r, "\chi_G"] \arrow[d, "\partial \flat 1"'] & G \arrow[d, "1"] \\
G_0 \flat G \arrow[r, "\xi"] \arrow[d, "1 \flat \partial"'] & G \arrow[d, "\partial"] \\
G_0 \flat G_0 \arrow[r, "\chi_{G_0}"'] & G_0
\end{tikzcd}$$

commutes, χ_X being the canonical conjugation action for the object X. The commutativity of the upper part is called Peiffer condition, the commutativity of the lower part is called precrossed module condition.

A morphism $P\colon \mathbb{H} \to \mathbb{G}$ of crossed modules is given by morphisms $p\colon H \to G$ and $p_0\colon H_0 \to G_0$ such that the diagram

$$\begin{tikzcd}
H_0 \flat H \arrow[r, "p_0 \flat p"] \arrow[d, "\xi"'] & G_0 \flat G \arrow[d, "\xi"] \\
H \arrow[r, "p"] \arrow[d, "\partial"'] & G \arrow[d, "\partial"] \\
H_0 \arrow[r, "p_0"'] & G_0
\end{tikzcd}$$

commutes. In the following, we will refer to the upper commutative square above by saying that the pair (p, p_0) is equivariant with respect to the actions.

Internal crossed modules with their morphisms form a category denoted by $\mathit{XMod}(\mathcal{C})$.

Remark 2.4. In an arbitrary semi-abelian category, the notion of crossed module introduced by G. Janelidze in [31] is stronger than the one we adopt here. The notion we use (already considered in [31] and further studied in [37]) is equivalent to the original one thanks to the condition "Huq = Smith", as proved in [39].

In the following proposition we consider $\mathit{Grpd}(\mathcal{C})$ as a category, that is, we forget natural transformations.

Proposition 2.5. *(Janelidze [31]) The categories $\mathit{Grpd}(\mathcal{C})$ and $\mathit{XMod}(\mathcal{C})$ are equivalent.*

PROOF. (sketch) Let

$$\begin{tikzcd}
G_1 \times_{c,d} G_1 \arrow[r, "m"] & G_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G_0 \arrow[l, "e" description]
\end{tikzcd}$$

be a groupoid and consider the following commutative diagram, where the rows are kernel diagrams:

$$\begin{tikzcd} G_0\flat G \arrow[r, "j_{G_0,G}"] \arrow[d, "\exists!"', "\xi"] & G_0+G \arrow[r, "{[1,0]}"] \arrow[d, "{[e,g]}"] & G_0 \arrow[d, "1"] \\ G \arrow[r, "g"'] & G_1 \arrow[r, "c"'] & G_0 \end{tikzcd} \tag{1}$$

We obtain a crossed module

$$G \xrightarrow{g} G_1 \xrightarrow{d} G_0\,, \qquad G_0\flat G \xrightarrow{\xi} G.$$

This describes the equivalence functor

$$J\colon\, Grpd(\mathcal{C}) \to XMod(\mathcal{C})$$

on objects; its extension to arrows is straightforward.

Conversely, let

$$G \xrightarrow{\partial} G_0\,, \qquad G_0\flat G \xrightarrow{\xi} G$$

be a crossed module and consider the semi-direct product, given by the coequalizer q_ξ below

$$\begin{tikzcd} G_0\flat G \arrow[rr, "j_{G_0,G}"] \arrow[dr, "\xi"'] & & G_0+G \arrow[r, "q_\xi"] & G\rtimes_\xi G_0\,. \\ & G \arrow[ur, "i_G"'] & & \end{tikzcd}$$

We obtain a reflexive graph

$$\begin{tikzcd} G_1 = \; G\rtimes_\xi G_0 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G_0\,, \arrow[l, "e" description] \end{tikzcd}$$

where c and $e = q_\xi \cdot i_{G_0}$ are, respectively, the canonical projection from and the canonical injection into the semi-direct product, and d is the unique morphism such that the diagram

$$\begin{tikzcd} G \arrow[r, "g"] \arrow[dr, "\partial"'] & G\rtimes_\xi G_0 \arrow[d, "d"] & G_0 \arrow[l, "e"'] \arrow[dl, "1"] \\ & G_0 & \end{tikzcd}$$

commutes, with $g = q_\xi \cdot i_G$ the canonical injection of G into the semi-direct product. For a detailed proof, see [31].

We will refer to the functor J as to the *normalization functor*, and to its quasi-inverse as to the *denormalization functor*. When no confusion arises, we will identify a groupoid with the corresponding crossed module writing $\mathbb{G}$ instead of $J(\mathbb{G})$, and ignoring in the same way the quasi-inverse of J.

Remark 2.6. 1. In [26] it is mentioned that, under the equivalence $\mathit{Grpd}(\mathcal{C}) \simeq \mathit{XMod}(\mathcal{C})$, a morphism $P\colon \mathbb{H} \to \mathbb{G}$ of crossed modules corresponds to a weak equivalence if and only if the arrows induced on kernels and cokernels are isomorphisms:

$$\begin{tikzcd}
\operatorname{Ker}\partial \arrow[r, "\simeq"] \arrow[d] & \operatorname{Ker}\partial \arrow[d] \\
H \arrow[r, "p"] \arrow[d, "\partial"'] & G \arrow[d, "\partial"] \\
H_0 \arrow[r, "p_0"] \arrow[d] & G_0 \arrow[d] \\
\operatorname{Coker}\partial \arrow[r, "\simeq"] & \operatorname{Coker}\partial
\end{tikzcd}$$

Such a morphism will be called a weak equivalence of crossed modules.

2. It is easy to show that a morphism $P\colon \mathbb{H} \to \mathbb{G}$ of crossed modules corresponds to a discrete fibration of groupoids if and only if $p\colon H \to G$ is an isomorphism. Such morphisms will be called *discrete fibrations of crossed modules.*

Notation 2.7. Here and in the following we will denote kernels of the codomain arrows with the lower case letter of the groupoid involved, e.g. the following sequence is short exact:

$$G \xrightarrow{g} G_1 \xrightarrow{c} G_0 \,.$$

Moreover, the composite $i \cdot g$ provides a kernel of the domain arrow d; we will often write $g^\bullet$ for $i \cdot g$.

The category $\mathit{XMod}(\mathcal{C})$ has an obvious 2-categorical structure. In fact, it suffices to translate the notion of natural transformation for internal functors into the language of crossed modules in order to obtain the 2-cells of $\mathit{XMod}(\mathcal{C})$. Using protomodularity, we get the following simpler notion, called transformation of Peiffer graphs in [37].

Definition 2.8. Consider two parallel morphisms $P, Q\colon \mathbb{H} \rightrightarrows \mathbb{G}$ of crossed modules. An arrow $\alpha\colon H_0 \to G_1 = G \rtimes_\xi G_0$ is a *natural transformation* between P and Q if $d \cdot \alpha = p_0$, $c \cdot \alpha = q_0$, and the diagram

$$\begin{tikzcd}
H \arrow[r, "\partial"] \arrow[d, "\langle p,q\rangle"'] & H_0 \arrow[d, "\alpha"] \\
G \times G \arrow[r, "m_0"'] & G_1
\end{tikzcd}$$

commutes, where $m_0 = g\sharp g^\bullet$ is the cooperator of the arrows

$$G \xrightarrow{g} G_1 \xleftarrow{g^\bullet} G$$

(see Section 9.2).

Set-theoretically, m_0 is the morphism that sends the pair of arrows $(a\colon x \to 0, b\colon y \to 0)$ to the composition $b^{-1} \cdot a\colon x \to y$; in other words, m_0 performs the division a/b.

Lemma 2.9. *Arrows satisfying Definition 2.8 correspond bijectively to natural transformations between the internal functors determined by the morphisms P and Q.*

Proof. Observe first that the diagram in Definition 2.8 commutes if and only if

$$\begin{tikzcd} H \arrow[r, "p"] \arrow[d, "\langle \alpha\cdot\partial, g\cdot q\rangle"'] \arrow[dr, phantom, "(i)"] & G \arrow[d, "g"] \\ G_1 \times_{c,d} G_1 \arrow[r, "m"'] & G_1 \end{tikzcd}$$

commutes. For the proof, it suffices to compute with elements and then apply the Yoneda embedding.

Now recall (for instance, from [7]) that a natural transformation between two internal functors $P = (p_1, p_0)$ and $Q = (q_1, q_0)\colon\ \mathbb{H} \to \mathbb{G}$ is defined as a morphism $\alpha\colon H_0 \to G_1$ satisfying $d \cdot \alpha = p_0$, $c \cdot \alpha = q_0$, and such that the diagram

$$\begin{tikzcd} H_1 \arrow[r, "\langle p_1, \alpha\cdot c\rangle"] \arrow[d, "\langle \alpha\cdot d, q_1\rangle"'] \arrow[dr, phantom, "(ii)"] & G_1 \times_{c,d} G_1 \arrow[d, "m"] \\ G_1 \times_{c,d} G_1 \arrow[r, "m"'] & G_1 \end{tikzcd}$$

commutes. So we have to prove that (i) commutes if and only if (ii) commutes. The "if" part is dealt with by precomposing the diagram (ii) with the monomorphism $h\colon H \to H_1$. Conversely, since the base category is protomodular, the pair (h, e) is (strongly) jointly epic, so that (ii) commutes if and only if it commutes when precomposed both with h and e. The first precomposition is precisely (i), the second one is trivial.

In conclusion we have proved the following (quite tautological)

Corollary 2.10. *The equivalence between Grpd($\mathcal{C}$) and XMod($\mathcal{C}$) extends to a biequivalence.*

Lemma 2.11. *Let $(\partial\colon H \to H_0,\ \xi\colon H_0\flat H \to H)$ be a crossed module and consider a morphism $\sigma\colon E \to H_0$. Consider also the pullback*

$$\begin{tikzcd} E \times_{\sigma,\partial} H \arrow[r, "\overline{\sigma}"] \arrow[d, "\overline{\partial}"'] \arrow[dr, phantom, "(i)"] & H \arrow[d, "\partial"] \\ E \arrow[r, "\sigma"'] & H_0 \end{tikzcd}$$

1. *The object $E \times_{\sigma,\partial} H$ can be equipped with a canonical action*

$$\overline{\xi}\colon E\flat(E \times_{\sigma,\partial} H) \to E \times_{\sigma,\partial} H$$

in such a way that the pair $(\overline{\partial}, \overline{\xi}\,)$ is a crossed module and the diagram (i) is a morphism of crossed modules.

2. *Moreover, if $\sigma\colon E \to H_0$ is a regular epimorphism, then (i) is a weak equivalence.*

Proof. This is a crossed-module version of a standard fact about internal categories (see [20]), so we just sketch the proof.

1. The canonical action $\overline{\xi}$ is the factorization of the diagram

$$\begin{tikzcd}
E\flat(E \times_{\sigma,\partial} H) \arrow[r, "\sigma\flat\overline{\sigma}"] \arrow[d, "1\flat\overline{\partial}"'] & H_0\flat H \arrow[r, "\xi"] & H \arrow[d, "\partial"] \\
E\flat E \arrow[r, "\chi_E"'] & E \arrow[r, "\sigma"'] & H_0
\end{tikzcd}$$

(which commutes by naturality of χ and the precrossed module condition on (∂, ξ)) through the pullback (i).

2. Since (i) is a pullback, kernels of parallel arrows are isomorphic. Henceforth, since σ and $\overline{\sigma}$ are regular epimorphisms, (i) is also a pushout, so that the induced arrow $\operatorname{Coker}\overline{\partial} \to \operatorname{Coker}\partial$ is an isomorphism. By Remark 2.6, we conclude that (i) is a weak equivalence.

3. The bicategory of butterflies

In this section, we describe the bicategory $\mathcal{B}(\mathcal{C})$ of crossed modules and butterflies in $\mathcal{C}$.

3.1. Butterflies

The notion of butterfly has been introduced in the category of groups by B. Noohi in [40], see also [2] (a special case of butterflies was used by D. F. Holt in [29] to classify group extensions).

Definition 3.1. Let $\mathbb{G}$ and $\mathbb{H}$ be crossed modules. A *butterfly* from $\mathbb{H}$ to $\mathbb{G}$ is given by a commutative diagram of the form

$$\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
H_0 & & G_0
\end{tikzcd}$$

such that

1. $\rho \cdot \kappa = 0$, i.e. (κ, ρ) is a complex;
2. $\iota = \ker\sigma$ and $\sigma = \operatorname{coker}\iota$, i.e. (ι, σ) is a short exact sequence;
3. the diagram

$$\begin{tikzcd}
E\flat H \arrow[r, "\sigma\flat 1"] \arrow[d, "1\flat\kappa"'] & H_0\flat H \arrow[r, "\xi"] & H \arrow[d, "\kappa"] \\
E\flat E \arrow[rr, "\chi_E"'] & & E
\end{tikzcd}$$

commutes, i.e. the pair $(\kappa, \xi \cdot (\sigma\flat 1))$ is a precrossed module,

4. the diagram

$$\begin{tikzcd} E\flat G \arrow[r, "\rho\flat 1"] \arrow[d, "1\flat\iota"'] & G_0\flat G \arrow[r, "\xi"] & G \arrow[d, "\iota"] \\ E\flat E \arrow[rr, "\chi_E"'] & & E \end{tikzcd}$$

commutes, i.e. the pair $(\iota, \xi \cdot (\rho\flat 1))$ is a precrossed module.

Given the butterflies $E, E' \colon \mathbb{H} \to \mathbb{G}$, a *morphism of butterflies* is a morphism $f \colon E \to E'$ such that the diagrams

$$\begin{tikzcd} & E \arrow[dd, "f"] \arrow[dr, "\rho"] & \\ H \arrow[ur, "\kappa"] \arrow[dr, "\kappa'"'] & & G_0 \\ & E' \arrow[ur, "\rho'"'] & \end{tikzcd} \qquad \begin{tikzcd} & E \arrow[dd, "f"] \arrow[dl, "\sigma"'] & \\ H_0 & & G \arrow[ul, "\iota"'] \arrow[dl, "\iota'"] \\ & E' \arrow[ul, "\sigma'"] & \end{tikzcd}$$

commute.

When no confusion is expected, we denote a butterfly $(E, \kappa, \rho, \iota, \sigma)$ from $\mathbb{H}$ to $\mathbb{G}$ simply by

$$E \colon \mathbb{H} \to \mathbb{G}\,.$$

Observe that, since (ι, σ) is a short exact sequence, the situation is not as symmetrical as it may appear at first sight. Actually ι is a mono, so the action $\xi \cdot (\rho\flat 1)$ is nothing but the conjugation action χ_E restricted to the subobject G. Moreover ι can be recovered as the normalization of the equivalence relation $(R_\sigma, \sigma_1, \sigma_2)$, i.e. $\iota = \sigma_1 \cdot \ker(\sigma_2)$. Observe also that, if $f \colon E \to E'$ is a morphism of butterflies, then in particular f is a morphism of short exact sequences, so that, by the short five lemma (see Section 9.1), it is an isomorphism.

Remark 3.2. Conditions 3 and 4 in Definition 3.1 imply that the pairs $(\kappa, \xi \cdot (\sigma\flat 1))$ and $(\iota, \xi \cdot (\rho\flat 1))$ are indeed crossed modules, and that

$$\begin{tikzcd} H \arrow[d, "\partial"'] & H \arrow[l, "1"'] \arrow[d, "\kappa"] \\ H_0 & E \arrow[l, "\sigma"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "1"] \arrow[d, "\iota"'] & G \arrow[d, "\partial"] \\ E \arrow[r, "\rho"'] & G_0 \end{tikzcd}$$

are morphisms of crossed modules, hence discrete fibrations by Remark 2.6.

3.2. Fractors

Using the equivalence between crossed modules and groupoids described in Proposition 2.5, butterflies correspond to fractors.

Definition 3.3. Let $\mathbb{H}$ and $\mathbb{G}$ be groupoids. A *fractor* from $\mathbb{H}$ to $\mathbb{G}$ is a diagram of the form

$$\begin{tikzcd}
& R \arrow[dl, "\overline{\sigma}"'] \arrow[dr, shift left, "d"] \arrow[dr, shift right, "c"'] & & R_\sigma \arrow[dl, shift right, "\sigma_1"'] \arrow[dl, shift left, "\sigma_2"] \arrow[dr, "\overline{\rho}"] & \\
H_1 \arrow[dr, shift left, "d"] \arrow[dr, shift right, "c"'] & & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & & G_1 \arrow[dl, shift right, "d"'] \arrow[dl, shift left, "c"] \\
& H_0 & & G_0 &
\end{tikzcd}$$

where

1. σ is a regular epimorphism, and R_σ is its kernel pair;
2. ρ coequalizes $d, c\colon R \rightrightarrows E$;
3. $(\overline{\sigma}, \sigma)$ and $(\overline{\rho}, \rho)$ are discrete fibrations.

Let us explain now how to construct a fractor from a butterfly and vice-versa. Denormalizing the morphisms of crossed modules in the diagram

$$\begin{tikzcd}
& H \arrow[dl, "1"'] \arrow[dr, "\kappa"] & & G \arrow[dl, "\iota"'] \arrow[dr, "1"] & \\
H \arrow[dr, "\partial"'] & & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & & G \arrow[dl, "\partial"] \\
& H_0 & & G_0 &
\end{tikzcd}$$

one easily gets a fractor as above, where $H_1 = H \rtimes_{\xi_H} H_0$, $G_1 = G \rtimes_{\xi_G} G_0$ and $R = H \rtimes_{\xi_H \cdot \sigma\flat 1} E$. The fact that the groupoid associated with ι is isomorphic to $(R_\sigma, \sigma_1, \sigma_2)$ is due to the fact that $\iota = \ker \sigma$. Finally, ρ coequalizes d and c since the pair

$$H \xrightarrow{\langle h,0\rangle} R \xleftarrow{e} E$$

is jointly (strongly) epic, by protomodularity.

Conversely, starting from a fractor as above, we get the butterfly

$$\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr, "d\cdot\langle h,0\rangle"] & & \operatorname{Ker}\sigma \simeq G \arrow[dl, "j"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
H_0 & & G_0
\end{tikzcd}$$

where $\langle h, 0\rangle\colon H \to R$ comes from the universal property of the pullback

$$\begin{tikzcd}
H_1 \arrow[d, "c"'] & R \arrow[l, "\overline{\sigma}"'] \arrow[d, "c"] \\
H_0 & E \arrow[l, "\sigma"]
\end{tikzcd}$$

and the isomorphism $\operatorname{Ker}\sigma \simeq G$ is the composite of the following isomorphisms, determined by the bottom pullback squares:

$$
\begin{tikzcd}
\operatorname{Ker}\sigma \arrow[d, "j"'] & \operatorname{Ker}\sigma_2 \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & G \arrow[d, "g"] \\
E \arrow[d, "\sigma"'] & R_\sigma \arrow[l, "\sigma_1"'] \arrow[r, "\overline{\rho}"] \arrow[d, "\sigma_2"] & G_1 \arrow[d, "c"] \\
H_0 & E \arrow[l, "\sigma"] \arrow[r, "\rho"'] & G_0
\end{tikzcd}
$$

Remark 3.4. Given a fractor as in Definition 3.3, one can consider also the kernel pair of the map $\overline{\sigma}$, and perform the construction below, where the dashed arrows are suitably obtained by the universal property of the kernel pair R_σ:

$$
\begin{tikzcd}
 & & & R_{\overline{\sigma}} \arrow[dl, shift right, "\overline{\sigma}_1"'] \arrow[dl, shift left, "\overline{\sigma}_2"] \arrow[dr, dashed, shift left, "\overline{d}"] \arrow[dr, dashed, shift right, "\overline{c}"'] & & & \\
 & & R \arrow[dl, "\overline{\sigma}"'] \arrow[dr, shift left, "d"] \arrow[dr, shift right, "c"'] & & R_\sigma \arrow[dl, shift right, "\sigma_1"'] \arrow[dl, shift left, "\sigma_2"] \arrow[dr, "\overline{\rho}"] & & \\
 & H_1 \arrow[dr, shift left, "d"] \arrow[dr, shift right, "c"'] & & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & & G_1 \arrow[dl, shift right, "d"'] \arrow[dl, shift left, "c"] & \\
 & & H_0 & & G_0 & &
\end{tikzcd}
$$

One finds out that the central square is a double groupoid over E. More precisely, it is a centralizing double groupoid, as defined by D. Bourn in [15], since $(\overline{\sigma}, \sigma)$ is a discrete fibration. Together with the two other squares, this gives rise to a particular profunctor $\mathbb{H} \nrightarrow \mathbb{G}$ of groupoids, independently studied by D. Bourn in [15] (profunctors were introduced by J. Bénabou with the name of distributeurs [6], an internal version can be found in [35]). Indeed, the correspondence between butterflies and fractors described above is part of a biequivalence between the bicategory of butterflies and a suitable sub-bicategory of the bicategory of profunctors, as explained in [38].

3.3. Identity butterflies

The canonical fractor associated with a groupoid gives the identity butterfly associated with a crossed module. In order to construct explicitly the identity butterfly on a crossed module $\mathbb{G}$, we consider the groupoid associated with $\mathbb{G}$, as in the proof of Proposition 2.5

$$
\begin{tikzcd}
G_1 = G \rtimes_\xi G_0 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G_0 \arrow[l, "e" description]
\end{tikzcd} .
$$

Following Example 2.2, one can associate with $\mathbb{G}$ the fractor

$$
\begin{tikzcd}
& R_c \arrow[dl, "\overline{d}"'] \arrow[dr, shift left, "c_1"] \arrow[dr, shift right, "c_2"'] & & R_d \arrow[dl, shift right, "d_1"'] \arrow[dl, shift left, "d_2"] \arrow[dr, "\overline{c}"] & \\
G_1 \arrow[dr, shift left, "d"] \arrow[dr, shift right, "c"'] & & G_1 \arrow[dl, "d"] \arrow[dr, "c"'] & & G_1 \arrow[dl, shift right, "d"'] \arrow[dl, shift left, "c"] \\
& G_0 & & G_0 &
\end{tikzcd}
$$

The corresponding butterfly, by means of the normalization process described in Section 3.2, is called the identity butterfly of the crossed module $\mathbb{G}$. Actually it acts as an identity with respect to the composition that will be introduced in Section 3.4. It is represented explicitly in the diagram below:

$$
\begin{tikzcd}
G \arrow[dd, "\partial"'] \arrow[dr, "g"] & & G \arrow[dl, "g^\bullet"'] \arrow[dd, "\partial"] \\
& G_1 \arrow[dl, "d"] \arrow[dr, "c"'] & \\
G_0 & & G_0
\end{tikzcd}
$$

In fact, in this paper we will use as identity butterfly the isomorphic contravariant version of the one above:

$$
\begin{tikzcd}
G \arrow[dd, "\partial"'] \arrow[dr, "g^\bullet"] & & G \arrow[dl, "g"'] \arrow[dd, "\partial"] \\
& G_1 \arrow[dl, "c"] \arrow[dr, "d"'] & \\
G_0 & & G_0
\end{tikzcd}
$$

the isomorphism being realized by the inverse map $i\colon G_1 \to G_1$. This choice does not affect the computations, and it is coherent with the normalization of a groupoid via the kernel of the codomain.

Remark 3.5. The fact that we can choose between these two butterflies for the identity is just an instance of the following more general fact, which will be useful in proving Theorem 5.6.
Let a crossed module $\partial\colon G \to G_0$ be given, and let us consider the canonical identity butterfly $(G_1, g^\bullet, d, g, c)$. For any other (isomorphic) groupoid representation of the crossed module

$\partial\colon G \to G_0$, we obtain, via normalization, a butterfly

$$
\begin{tikzcd}
G \arrow[dd, "\partial"'] \arrow[dr, "\bar{g}^{\bullet}"] & & G \arrow[dd, "\partial"] \arrow[dl, "\bar{g}"'] \\
& \overline{G}_1 \arrow[dl, "\bar{c}"] \arrow[dr, "\bar{d}"'] & \\
G_0 & & G_0
\end{tikzcd}
$$

isomorphic to the canonical one.

3.4. Composition of butterflies

Let us consider the butterflies E and E':

$$\mathbb{H} \xrightarrow{E} \mathbb{G} \xrightarrow{E'} \mathbb{K}$$

We are going to define their composition. To this end, let us consider the diagram

$$
\begin{tikzcd}
& & & Q \arrow[dddllll, bend right, "\overline{\sigma\cdot r}"'] \arrow[dddrrrr, bend left, "\overline{\rho'\cdot s}"] & & & \\
& & & E\times_{\rho,\sigma'} E' \arrow[u, "q"] \arrow[ddl, "r"'] \arrow[ddr, "s"] & & & \\
H \arrow[dd, "\partial"'] \arrow[urrr, "\langle\kappa,0\rangle"] \arrow[dr, "\kappa"] & & & G \arrow[u, "\langle\iota,\kappa'\rangle"] \arrow[dl, "\iota"] \arrow[dd, "\partial"] \arrow[dr, "\kappa'"'] & & & K \arrow[dd, "\partial"] \arrow[ulll, "\langle 0,\iota'\rangle"'] \arrow[dl, "\iota'"'] \\
& E \arrow[dl, "\sigma"] \arrow[drr, "\rho"'] & & & & E' \arrow[dll, "\sigma'"] \arrow[dr, "\rho'"'] & \\
H_0 & & & G_0 & & & K_0
\end{tikzcd}
$$

where

- $E \times_{\rho,\sigma'} E'$ is the pullback of ρ and σ', with projections r and s, so that

$s\cdot\langle\kappa,0\rangle = 0,\ r\cdot\langle 0,\iota'\rangle = 0,\ r\cdot\langle\kappa,0\rangle = \kappa,\ s\cdot\langle 0,\iota'\rangle = \iota',\ r\cdot\langle\iota,\kappa'\rangle = \iota,\ s\cdot\langle\iota,\kappa'\rangle = \kappa',$

- (Q,q) is the cokernel of $\langle\iota,\kappa'\rangle$,
- $\overline{\sigma\cdot r}$ and $\overline{\rho'\cdot s}$ are defined by $\overline{\sigma\cdot r}\cdot q = \sigma\cdot r,\ \overline{\rho'\cdot s}\cdot q = \rho'\cdot s.$

Lemma 3.6. *With the previous notation, the diagram*

$$
\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr, "q\cdot\langle\kappa,0\rangle"] & & K \arrow[dd, "\partial"] \arrow[dl, "q\cdot\langle 0,\iota'\rangle"'] \\
& Q \arrow[dl, "\overline{\sigma\cdot r}"] \arrow[dr, "\overline{\rho'\cdot s}"'] & \\
H_0 & & K_0
\end{tikzcd}
$$

is a butterfly from $\mathbb{H}$ *to* $\mathbb{K}$. *It defines the composition of* E *and* E'.

Proof. We check that the previous diagram is indeed a butterfly from $\mathbb{H}$ to $\mathbb{K}$.

Commutativity of wings and condition 3.1.1 are easy to check.

Condition 3.1.2: first observe that $\overline{\sigma \cdot r}$ is a regular epimorphism (because σ and σ' are), so that it is enough to show that $q \cdot \langle 0, \iota' \rangle$ is the kernel of $\overline{\sigma \cdot r}$. Since ι' is the kernel of σ' and r is a pullback of σ', $\langle 0, \iota' \rangle$ is the kernel of r. Consider now the commutative diagram

$$\begin{tikzcd}
G & G \arrow[l, "1"'] \arrow[d, "{\langle \iota, \kappa' \rangle}"] & \\
E \arrow[d, "\sigma"'] \arrow[u, leftarrow, "\iota"] \arrow[dr, phantom, "(i)"] & E \times_{\rho,\sigma'} E' \arrow[l, "r"'] \arrow[d, "q"] & K = \mathrm{Ker}\, r \arrow[l, "{\langle 0, \iota' \rangle}"'] \arrow[d, "f"] \\
H_0 & Q \arrow[l, "\overline{\sigma \cdot r}"] & \mathrm{Ker}\,(\overline{\sigma \cdot r}) \arrow[l]
\end{tikzcd}$$

where f is induced by the universal property of $\mathrm{Ker}\,(\overline{\sigma \cdot r})$. It suffices to prove that (i) is a pullback. Indeed, if (i) is a pullback, then f is an isomorphism and, therefore, $q \cdot \langle 0, \iota' \rangle$ is the kernel of $\overline{\sigma \cdot r}$. Since q and σ are regular epimorphisms and ι is the kernel of σ, to show that (i) is a pullback is equivalent to showing that $\langle \iota, \kappa' \rangle$ is the kernel of q. Since $\langle \iota, \kappa' \rangle$ is a monomorphism (because ι is), to prove that $\langle \iota, \kappa' \rangle$ is a kernel (of its cokernel q) is equivalent to proving that $\langle \iota, \kappa' \rangle$ is closed under conjugation in $E \times_{\rho,\sigma'} E'$ (see [34, 37]). The action of $E \times_{\rho,\sigma'} E'$ on G is given by

$$(E \times_{\rho,\sigma'} E')\flat G \xrightarrow{r\flat 1} E\flat G \xrightarrow{\rho\flat 1} G_0\flat G \xrightarrow{\xi} G$$

or, equivalently, by

$$(E \times_{\rho,\sigma'} E')\flat G \xrightarrow{s\flat 1} E'\flat G \xrightarrow{\sigma'\flat 1} G_0\flat G \xrightarrow{\xi} G\,.$$

To prove the normality of $\langle \iota, \kappa' \rangle$ in $E \times_{\rho,\sigma'} E'$, it is enough to prove that the diagram

$$\begin{tikzcd}
(E \times_{\rho,\sigma'} E')\flat G \arrow[r, "r\flat 1"] \arrow[d, "1\flat\langle \iota, \kappa' \rangle"'] & E\flat G \arrow[r, "\rho\flat 1"] & G_0\flat G \arrow[r, "\xi"] & G \arrow[d, "{\langle \iota, \kappa' \rangle}"] \\
(E \times_{\rho,\sigma'} E')\flat(E \times_{\rho,\sigma'} E') \arrow[rrr, "\chi"'] & & & E \times_{\rho,\sigma'} E'
\end{tikzcd}$$

commutes. For this, compose with the pullback projections r and s and use the naturality of χ and, respectively, condition 3.1.3 on ι and condition 3.1.4 on κ'.

Condition 3.1.3: since

$$q\flat 1 \colon (E \times_{\rho,\sigma'} E')\flat H \to Q\flat H$$

is a (regular) epimorphism (see [37]), condition 3.1.3 follows from the commutativity of the whole diagram below

$$\begin{tikzcd}
(E\times_{\rho,\sigma'}E')\flat H \arrow[r, "q\flat 1"] \arrow[d, "1\flat\langle\kappa,0\rangle"'] & Q\flat H \arrow[r, "\overline{\sigma\cdot r}\flat 1"] & H_0\flat H \arrow[r, "\xi"] & H \arrow[d, "\langle\kappa,0\rangle"] \\
(E\times_{\rho,\sigma'}E')\flat(E\times_{\rho,\sigma'}E') \arrow[rrr, "\chi"'] \arrow[d, "q\flat q"'] & & & E\times_{\rho,\sigma'}E' \arrow[d, "q"] \\
Q\flat Q \arrow[rrr, "\chi"'] & & & Q
\end{tikzcd}$$

The lower rectangle commutes by naturality of χ. For the commutativity of the upper rectangle, compose with the pullback projections: composed with s, both composites become zero; as far as r is concerned, use condition 3.1.3 on κ.

Condition 3.1.4: same argument as for condition 3.1.3.

Proposition 3.7. *We have a bicategory*

$$\mathcal{B}(\mathcal{C})$$

with internal crossed modules as objects, butterflies as 1-cells, and morphisms of butterflies as 2-cells.

Proof. Composition of butterflies and identity butterflies have been described in Sections 3.4 and 3.3. The rest of the proof is long but straightforward.

Observe that in the identity butterfly (Section 3.3) both diagonals are short exact sequences. Butterflies with this property are called *flippable* (see [40]).

Proposition 3.8. *A flippable butterfly $E\colon \mathbb{H}\to\mathbb{G}$ is an equivalence in the bicategory $\mathcal{B}(\mathcal{C})$. A quasi-inverse $E^*\colon \mathbb{G}\to\mathbb{H}$ is obtained by switching the two wings of E.*

Proof. By symmetry, it is sufficient to compute $E\cdot E^*$, by taking the kernel pair R_σ of σ and then the cokernel of $\langle\kappa,\kappa\rangle$:

$$\begin{tikzcd}
& R_\sigma \arrow[ddl, "\sigma_1"'] \arrow[ddr, "\sigma_2"] & \\
& H \arrow[u, "\langle\kappa,\kappa\rangle"] \arrow[dl, "\kappa"] \arrow[dr, "\kappa"'] & \\
E \arrow[dr, "\sigma"'] & & E \arrow[dl, "\sigma"] \\
& H_0 &
\end{tikzcd}$$

Keeping in mind Section 3.2, consider the (isomorphic) kernels in the right discrete fibration in the fractor corresponding to E :

$$\begin{tikzcd} H \arrow[r, "\langle\kappa,\kappa\rangle"] \arrow[d, "1"'] & R_\sigma \arrow[r, "\bar{\rho}"] \arrow[d, shift right, "\sigma_1"'] \arrow[d, shift left, "\sigma_2"] & G_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ H \arrow[r, "\kappa"'] & E \arrow[r, "\rho"'] & G_0 \end{tikzcd}$$

The horizontal rows are exact, and this concludes the proof.

4. Butterflies and morphisms of crossed modules

In order to prove that $\mathcal{B}(\mathcal{C})$ is the bicategory of fractions of $Grpd(\mathcal{C})$ with respect to weak equivalences (Theorem 5.6), we have to construct a homomorphism of bicategories

$$\mathcal{F}\colon\ Grpd(\mathcal{C}) \to \mathcal{B}(\mathcal{C})\,.$$

This task will be completed only in Section 5.3, since we have to provide first some necessary constructions.

4.1. Split butterflies

A preliminary step consists in associating a split butterfly with any morphism of crossed modules.

Definition 4.1. A butterfly $E\colon \mathbb{H} \to \mathbb{G}$ is *split* when the short exact sequence

$$H_0 \xleftarrow{\ \sigma\ } E \xleftarrow{\ \iota\ } G$$

is split, that is, when there exists $s\colon H_0 \to E$ such that $\sigma \cdot s = 1_{H_0}$.
A morphism of split butterflies is simply a morphism of butterflies, so that it need not commute with sections.

Let $P\colon \mathbb{H} \to \mathbb{G}$ be a morphism of crossed modules. We are going to construct a split butterfly $E_P\colon \mathbb{H} \to \mathbb{G}$. For this, consider the pullback

$$\begin{tikzcd} E_P \arrow[r, "\overline{p}"] \arrow[d, "\sigma_P"'] & G_1 \arrow[d, "c"] \\ H_0 \arrow[r, "p_0"'] & G_0 \end{tikzcd}$$

If $\xi\colon G_0\flat G \to G$ is the action corresponding to the split epi $c\colon G_1 \to G_0$, it is easy to show that

$$H_0\flat G \xrightarrow{p_0\flat 1} G_0\flat G \xrightarrow{\ \xi\ } G$$

is the action corresponding to the split epi $\sigma_P\colon E_P \to H_0$.

Lemma 4.2. *The diagram*

$$\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr, "\langle \partial, i\cdot g\cdot p\rangle"] & & G \arrow[dl, "\langle 0,g\rangle"'] \arrow[dd, "\partial"] \\
& E_P \arrow[dl, "\sigma_P"] \arrow[dr, "d\cdot\overline{p}"'] & \\
H_0 & & G_0
\end{tikzcd}$$

is a split butterfly $E_P \colon \mathbb{H} \to \mathbb{G}$.

Proof. Commutativity of the two wings is obvious.

Condition 3.1.1: one computes $d \cdot \overline{p} \cdot \langle \partial, i \cdot g \cdot p\rangle = d \cdot i \cdot g \cdot p = c \cdot g \cdot p = 0 \cdot p = 0$.

Condition 3.1.2: the top-right to bottom-left diagonal is a split extension, since it is the pullback of a split extension.

Condition 3.1.3: To check the commutativity of

$$\begin{tikzcd}
E_P\flat H \arrow[r, "\sigma_P\flat 1"] \arrow[d, "1\flat\langle\partial, i\cdot g\cdot p\rangle"'] & H_0\flat H \arrow[r, "\xi"] & H \arrow[d, "\langle\partial, i\cdot g\cdot p\rangle"] \\
E_P\flat E_P \arrow[rr, "\chi_{E_P}"'] & & E_P
\end{tikzcd}$$

compose with the pullback projections $\sigma_P \colon E_P \to H_0$ and $\overline{p} \colon E_P \to G_1$. When composing with σ_P, use the naturality of χ and the precrossed module condition on $\mathbb{H}$. When composing with $\overline{p}$, the commutativity of the resulting diagram easily reduces to condition 3.1.3 on the identity butterfly on $\mathbb{G}$.

Condition 3.1.4: to check the commutativity of

$$\begin{tikzcd}
E_P\flat G \arrow[r, "(d\cdot\overline{p})\flat 1"] \arrow[d, "1\flat\langle 0,g\rangle"'] & G_0\flat G \arrow[r, "\xi"] & G \arrow[d, "\langle 0,g\rangle"] \\
E_P\flat E_P \arrow[rr, "\chi_{E_P}"'] & & E_P
\end{tikzcd}$$

compose once again with the pullback projections.

We have just seen that every morphism $P \colon \mathbb{H} \to \mathbb{G}$ yields a split butterfly, namely E_P. Also the converse is true. Indeed, let

$$\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, shift left, "\sigma"] \arrow[dr, "\rho"] & \\
H_0 \arrow[ur, shift left, "s"] & & G_0
\end{tikzcd}$$

be a split butterfly. If we precompose the commutative diagram of condition 3.1.3 with the arrow $s\flat 1\colon H_0\flat H \to E\flat H$, we get the commutativity of

$$\begin{tikzcd}
H_0\flat H \arrow[r, "\xi"] \arrow[d, "s\flat\kappa"'] & H \arrow[d, "\kappa"] \\
E\flat E \arrow[r, "\chi_E"'] & E
\end{tikzcd}$$

From the universal property of the semi-direct product (see [31], Theorem 1.3), we obtain a unique arrow $\overline{\kappa}$ making commutative the diagram

$$\begin{tikzcd}
H \arrow[r, "h"] \arrow[dr, "\kappa"'] & H_1 \arrow[d, "\overline{\kappa}"] & H_0 \arrow[l, "e"'] \arrow[dl, "s"] \\
& E &
\end{tikzcd}$$

The requested morphism $P\colon \mathbb{H} \to \mathbb{G}$ is the one corresponding to the following internal functor (notation as in Definition 3.3, Δ is the diagonal):

$$\begin{tikzcd}
H_1 \arrow[r, "\overline{\kappa}"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & E \arrow[r, "\langle 1, s\cdot\sigma\rangle"] \arrow[d, dotted, shift right, "\sigma"'] & R_\sigma \arrow[r, "\overline{\rho}"] \arrow[d, dotted, shift right, "\sigma_2"'] & G_1 \arrow[r, "i"] \arrow[d, dotted, shift right, "c"'] & G_1 \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\
H_0 \arrow[r, "1"'] \arrow[u, "e" description] & H_0 \arrow[r, "s"'] \arrow[u, dotted, shift right, "s"'] & E \arrow[r, "\rho"'] \arrow[u, dotted, shift right, "\Delta"'] & G_0 \arrow[r, "1"'] \arrow[u, dotted, shift right, "e"'] & G_0 \arrow[u, "e" description]
\end{tikzcd}$$

(following [21], Proposition 2.1, it suffices to check that this is a morphism of reflexive graphs and, for this, use the dotted arrows).

4.2. Reduced composition

Given a morphism $Q\colon \mathbb{K} \to \mathbb{H}$ of crossed modules and a butterfly $E\colon \mathbb{H} \to \mathbb{G}$, we can turn Q into a split butterfly $E_Q\colon \mathbb{K} \to \mathbb{H}$ as in Section 4.1, and then compose E_Q with E using composition of butterflies described in Section 3.4. We describe here a somehow easier way to calculate $E \cdot E_Q$, which will be called *reduced composition* and denoted by $E \cdot_{rc} Q$. Starting from the situation described below,

$$\begin{tikzcd}
K \arrow[r, "q"] \arrow[dd, "\partial"'] & H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \\
& & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
K_0 \arrow[r, "q_0"'] & H_0 & & G_0
\end{tikzcd}$$

we consider the pullback

$$\begin{tikzcd}
E' \arrow[r, "q'"] \arrow[d, "\sigma'"'] & E \arrow[d, "\sigma"] \\
K_0 \arrow[r, "q_0"'] & H_0
\end{tikzcd} \tag{2}$$

and the arrows $\langle 0,\iota\rangle\colon G\to E'$ and $\langle\partial,\kappa\cdot q\rangle\colon K\to E'$. Then $E\cdot_{rc}Q$ is given by

$$\begin{tikzcd}
K \arrow[dd, "\partial"'] \arrow[dr, "\langle\partial,\kappa\cdot q\rangle"] & & G \arrow[dd, "\partial"] \arrow[dl, "\langle 0,\iota\rangle"'] \\
& E' \arrow[dl, "\sigma'"] \arrow[dr, "\rho\cdot q'"'] & \\
K_0 & & G_0
\end{tikzcd}$$

and it coincides with the butterfly $E\cdot E_Q$ (see the next lemma). In particular, if $I_{\mathbb{H}}\colon\mathbb{H}\to\mathbb{H}$ is the identity butterfly (Section 3.3), then $I_{\mathbb{H}}\cdot_{rc}Q$ is precisely the split butterfly E_Q as in Section 4.1.

Lemma 4.3. *With the previous notation,* $E\cdot E_Q=E\cdot_{rc}Q$.

Proof. Let us consider the following picture, where all the squares are pullbacks and moreover, the down-right square is the discrete fibration of Remark 3.4:

$$\begin{tikzcd}
& & K_0\times_{q_0,c}H_1\times_{d,\sigma}E \arrow[dl, "r"'] \arrow[dr, "\phi"] & & \\
& E_Q \arrow[dl, "\sigma_Q"'] \arrow[dr, "\overline{q}"] & & R \arrow[dl, "\overline{\sigma}"'] \arrow[dr, "\overline{d}"'] \arrow[dr, dashed, shift left, "\overline{c}"] & \\
K_0 \arrow[dr, "q_0"'] & & H_1 \arrow[dl, "c"] \arrow[dr, "d"'] \arrow[dr, dashed, shift left, "c"] & & E \arrow[dl, "\sigma"] \\
& H_0 & & H_0 &
\end{tikzcd}$$

By commutativity of limits, the topmost object is the limit over the W-shaped diagram $\{q_0,c,d,\sigma\}$, whence the notation adopted. The pullback (2) in Section 4.2 determines a unique

$$\omega\colon K_0\times_{q_0,c}H_1\times_{d,\sigma}E\to E'=K_0\times_{q_0,\sigma}E$$

such that $q'\cdot\omega=\overline{c}\cdot\phi$ and $\sigma'\cdot\omega=\sigma_Q\cdot r$. Now we can consider the diagram

$$\begin{tikzcd}
G \arrow[d, "\langle 0,\iota\rangle"'] \arrow[dr, phantom, "(i)"] & G \arrow[l, "1"'] \arrow[d, "\langle 0,0,\iota\rangle"] & \\
E' \arrow[d, "\sigma'"'] \arrow[dr, phantom, "(ii)"] & K_0\times_{q_0,c}H_1\times_{d,\sigma}E \arrow[l, "\omega"'] \arrow[d, "r"] \arrow[dr, phantom, "(iii)"] & H \arrow[l, "\langle 0,h,\kappa\rangle"'] \arrow[d, "1"] \\
K_0 & E_Q \arrow[l, "\sigma_Q"] & H \arrow[l, "\langle 0,h\rangle"]
\end{tikzcd}$$

By composing with pullback projections, one easily shows that (i) and (iii) commute, so that all the squares are commutative. Then, since r is a regular epimorphism, by Theorem 9.1, (ii) is a pullback square, hence ω is a regular epimorphism and it has the same kernel as σ_Q. Moreover, since $\ker \sigma_Q = \langle 0, h \rangle$, (iii) proves that $\ker \omega = \langle 0, h, \kappa \rangle$.

So far, we proved a technical

Lemma 4.4. *The sequence*

$$H \xrightarrow{\langle 0,h,\kappa \rangle} K_0 \times_{q_0,c} H_1 \times_{d,\sigma} E \xrightarrow{\ \omega\ } E'$$

is exact.

Now we can finally prove that $E \cdot E_Q = E \cdot_{rc} Q$. To this end, let us consider the following diagram

$$\begin{tikzcd}
 & & E' \arrow[ddddll, bend right, "\sigma'"'] \arrow[ddddrr, bend left, "\rho\cdot q'"] & & \\
 & & K_0 \times_{q_0,c} H_1 \times_{d,\sigma} E \arrow[u, "\omega"'] \arrow[ddl, "r"'] \arrow[ddr, "\overline{d}\cdot\phi"] & & \\
K \arrow[urr, "{\langle \partial, i\cdot h\cdot q, 0\rangle}"] \arrow[dr, "{\langle \partial, i\cdot h\cdot q\rangle}"] \arrow[dd, dashed] & & H \arrow[u, "{\langle 0,h,\kappa\rangle}"] \arrow[dl, "{\langle 0,h\rangle}"] \arrow[dr, "\kappa"] \arrow[dd, dashed] & & G \arrow[ull, "{\langle 0,0,\iota\rangle}"'] \arrow[dl, "\iota"'] \arrow[dd, dashed] \\
 & E_Q \arrow[dl, "\sigma_Q"] \arrow[dr, "d\cdot\overline{q}"'] & & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
K_0 & & H_0 & & G_0
\end{tikzcd}$$

The two butterflies are, from left to right, the split butterfly $E_Q \colon \mathbb{K} \to \mathbb{H}$ corresponding to the morphism Q, and $E \colon \mathbb{H} \to \mathbb{G}$. What we are to show is that the above diagram yields the composition of the two. In fact, the resulting butterfly would be precisely $E \cdot_{rc} Q$, as desired.

By composition of pullbacks, the square $d \cdot \overline{q} \cdot r = \sigma \cdot \overline{d} \cdot \phi$ above is a pullback, and by Lemma 4.4, ω is the cokernel of $\langle 0, h, \kappa \rangle$. Moreover σ' is (the only morphism) such that $\sigma' \cdot \omega = \sigma_Q \cdot r$ and $\rho \cdot q'$ is (the only one) such that $\rho \cdot q' \cdot \omega = \rho \cdot \overline{d} \cdot \phi$, and this concludes the proof.

The following statement will help us in defining the embedding of crossed modules into butterflies.

Proposition 4.5. *Reduced composition gives an action of crossed module morphisms on butterflies. This means that, for morphisms $P \colon \mathbb{K}' \to \mathbb{K}, Q \colon \mathbb{K} \to \mathbb{H}$ and butterflies $E \colon \mathbb{H} \to \mathbb{G}, F \colon \mathbb{G} \to \mathbb{G}'$, we have:*

1. $(F \cdot E) \cdot_{rc} Q \cong F \cdot (E \cdot_{rc} Q)\,,$
2. $E \cdot_{rc} (Q \cdot P) \cong (E \cdot_{rc} Q) \cdot_{rc} P\,,$

3. $E \cdot_{rc} I \cong E$.

PROOF. (sketch) The proof of 3 is trivial, that of 2 is straightforward. The proof of 1 can be easily deduced from the particular case

*1**. $F \cdot_{rc} Q \cong F \cdot (I \cdot_{rc} Q)$

where I is the identity butterfly on the domain of F. Actually one computes

$$(F \cdot E) \cdot_{rc} Q \cong (F \cdot E) \cdot (I \cdot_{rc} Q) \cong F \cdot (E \cdot (I \cdot_{rc} Q)) \cong F \cdot ((E \cdot I) \cdot_{rc} Q) \cong F \cdot (E \cdot_{rc} Q)\,.$$

Hence we are to prove 1^* holds, but since $I \cdot_{rc} Q = E_Q$, this is precisely the content of the proof of the consistency of reduced composition described in Section 4.2.

5. Butterflies are fractions

In this section we prove the main result of the paper, but first it is necessary to introduce the fractions the title refers to.

As for the case of groups (see [40]), given a butterfly, it is possible to construct a span of morphisms, one being a weak equivalence. By denormalizing, this yields a fraction of internal functors.

5.1. Bicategories of fractions

Categories of fractions have been introduced by P. Gabriel and M. Zisman in [28] in order to give a simplicial construction of the homotopy category of CW-complexes. In order to study toposes locally equivalent to toposes of sheaves on a topological space, in [44] D. Pronk generalized Gabriel-Zisman notion introducing bicategories of fractions.

Imitating the usual universal property of the category of fractions, it is clear how to state the universal property of the bicategory of fractions of a bicategory $\mathcal{B}$ with respect to a class Σ of 1-cells ([44]): the bicategory of fractions of $\mathcal{B}$ with respect to Σ is a homomorphism of bicategories

$$\mathcal{P}_\Sigma \colon \mathcal{B} \to \mathcal{B}[\Sigma^{-1}]$$

universal among all homomorphisms $\mathcal{F} \colon \mathcal{B} \to \mathcal{A}$ with $\mathcal{F}(S)$ an equivalence for all $S \in \Sigma$. This means that, for every bicategory $\mathcal{A}$,

$$- \cdot \mathcal{P}_\Sigma \;:\; \mathit{Hom}(\mathcal{B}[\Sigma^{-1}], \mathcal{A}) \to \mathit{Hom}_\Sigma(\mathcal{B}, \mathcal{A})$$

is a biequivalence of bicategories, where a homomorphism $\mathcal{F} \colon \mathcal{B} \to \mathcal{A}$ lies in $\mathit{Hom}_\Sigma(\mathcal{B}, \mathcal{A})$ when $\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$.

The real challenge with bicategories of fractions is to find an explicit, manageable description of $\mathcal{B}[\Sigma^{-1}]$. A first general result in this direction, established in [44], states that, if Σ satisfies some suitable conditions (has a "right calculus of fractions"), then the bicategory of fraction exists and can be described as follows: the objects of $\mathcal{B}[\Sigma^{-1}]$ are those of $\mathcal{B}$ and the 1-cells of $\mathcal{B}[\Sigma^{-1}]$ are spans of 1-cells in $\mathcal{B}$ with the backward leg in Σ (this is a non-straightforward generalization of a well-known result from [28]).

In order to prove that butterflies provide the bicategory of fractions of $Grpd(\mathcal{C})$ with respect to weak equivalences, we will use the following result. (The numeration "BFn" is meant to remind "Bicategory of Fractions".)

Proposition 5.1. *(Pronk [44]) Let Σ be a class of 1-cells in a bicategory $\mathcal{B}$. Assume that Σ has a right calculus of fractions and consider a homomorphism of bicategories $\mathcal{F}\colon \mathcal{B} \to \mathcal{A}$ such that*

BF0. $\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$;
BF1. $\mathcal{F}$ is surjective up to equivalence on objects;
BF2. $\mathcal{F}$ is full and faithful on 2-cells;
BF3. For every 1-cell F in $\mathcal{A}$ there exist 1-cells G and W in $\mathcal{B}$ with W in Σ and a 2-cell $\mathcal{F}(G) \Rightarrow F \cdot \mathcal{F}(W)$.

Then the (essentially unique) extension

$$\widehat{\mathcal{F}}\colon \mathcal{B}[\Sigma^{-1}] \to \mathcal{A}$$

of $\mathcal{F}$ through $\mathcal{P}_\Sigma$ is a biequivalence.

5.2. From butterflies to fractions

Before showing how a butterfly turns into a fraction, we need one more property of butterflies.

Lemma 5.2. *Consider a butterfly $E\colon \mathbb{H} \to \mathbb{G}$. The arrows*

$$\kappa : H \to E \leftarrow G : \iota$$

cooperate (see Section 9.2), that is, there exists a unique arrow $\varphi = \kappa\sharp\iota$ such that the diagram

$$\begin{tikzcd} H \arrow[r, "\langle 1,0\rangle"] \arrow[dr, "\kappa"'] & H \times G \arrow[d, "\varphi"'] & G \arrow[l, "\langle 0,1\rangle"'] \arrow[dl, "\iota"] \\ & E & \end{tikzcd}$$

commutes.

Proof. The fact that κ and ι cooperate is equivalent to the fact that the composition

$$G \otimes H \xrightarrow{\delta} G + H \xrightarrow{[\iota,\kappa]} E$$

is the zero morphism (see for example [37], where the symbol $\diamond$ is used), where δ is the diagonal of the pullback

$$\begin{tikzcd} G \otimes H \arrow[r, "\delta_2"] \arrow[d, "\delta_1"'] & G\flat H \arrow[d, "j_{G,H}"] \\ H\flat G \arrow[r, "j_{H,G}"'] & G + H \end{tikzcd}$$

The equation $[\iota,\kappa]\cdot\delta = 0$ follows from the commutativity of

$$\begin{tikzcd}
G\otimes H \arrow[r, "\delta_1"] \arrow[ddd, "\delta"'] & H\flat G \arrow[r, "j_{H,G}"] & G+H \arrow[r, "{[0,1]}"] & H\simeq 0\flat H \arrow[d, "!\flat 1"] \\
& & & H_0\flat H \arrow[d, "\xi"] \\
& & & H \arrow[d, "\kappa"] \\
G+H \arrow[rrr, "{[\iota,\kappa]}"'] & & & E
\end{tikzcd}$$

which can be reduced to the commutativity of

$$\begin{tikzcd}
E\flat H \arrow[r, "\sigma\flat 1"] \arrow[d, "j_{E,H}"'] & H_0\flat H \arrow[r, "\xi"] & H \arrow[d, "\kappa"] \\
E+H \arrow[rr, "{[1,\kappa]}"'] & & E
\end{tikzcd}$$

which itself follows from condition 3.1.3 using the equality

$$\chi_E = [1,1]\cdot j_{E,E}\colon E\flat E \to E+E \to E\,.$$

Remark 5.3. The fact that κ and ι cooperate may be used as a starting point for creating many non-trivial examples of butterfly: one starts by considering two cooperating normal subobjects and then computes their respective cokernels.

We established in Lemma 5.2 that the two crossed modules κ and ι cooperate. More is true:

Proposition 5.4. *Consider a butterfly $E\colon \mathbb{H}\to\mathbb{G}$. The cooperator φ of κ and ι is a crossed module, for a suitable action $\overline{\xi}$, and the diagram*

$$\begin{tikzcd}
H \arrow[d, "\partial"'] & H\times G \arrow[l, "\pi_H"'] \arrow[r, "\pi_G"] \arrow[d, "\varphi"'] \arrow[dl, phantom, "(i)"] \arrow[dr, phantom, "(ii)"] & G \arrow[d, "\partial"] \\
H_0 & E \arrow[l, "\sigma"] \arrow[r, "\rho"'] & G_0
\end{tikzcd}$$

is a span of crossed modules,

$$\begin{tikzcd}
\mathbb{H} & {[E]} \arrow[l, "{(\pi_H,\sigma)}"'] \arrow[r, "{(\pi_G,\rho)}"] & \mathbb{G}
\end{tikzcd}$$

with (π_H,σ) being a weak equivalence.

Proof. The commutativity of (i) and (ii) can be proved by precomposing with the jointly epimorphic pair

$$\langle 1,0\rangle : H \to H\times G \leftarrow G : \langle 0,1\rangle\,.$$

Moreover, (i) is a pullback because it is commutative and the regular epimorphisms π_H and σ have the same kernel (use Theorem 9.1). Therefore, we can apply Lemma 2.11 to (i) getting that φ is a crossed module and that (i) is a weak equivalence of crossed modules. The action $\overline{\xi}$ that makes φ a crossed module is the unique morphism such that $\pi_H\cdot\overline{\xi} = \xi\cdot(\sigma\flat\pi_H)$ and $\varphi\cdot\overline{\xi} = \chi_E\cdot(1\flat\varphi)$, see Lemma 2.11.
It remains to show that (ii) is a morphism of crossed modules, i.e. that the diagram

$$\begin{tikzcd}
E\flat(H\times G) \arrow[r, "\rho\flat\pi_G"] \arrow[d, "\bar\xi"'] \arrow[dr, phantom, "(iii)"] & G_0\flat G \arrow[d, "\xi"] \\
H\times G \arrow[r, "\pi_G"'] & G
\end{tikzcd}$$

commutes. For this, we need a different description of $\overline{\xi}$. Let us consider the following diagram:

$$\begin{tikzcd}
& H\times G \arrow[rr, "\pi_G"] \arrow[dl, dashed, "\bar h"'] \arrow[dd, "\pi_H"] \arrow[dr, phantom, "(iv)"] & & G \arrow[dl, "\iota"] \arrow[dd] \\
R \arrow[rr, "\bar c"] \arrow[dd, "\bar\sigma"'] & & E \arrow[dd, "\sigma"] & \\
& H \arrow[rr] \arrow[dl, "h"'] & & 0 \arrow[dl] \\
H_1 \arrow[rr, "c"'] & & H_0 &
\end{tikzcd}$$

All the squares of solid lines are pullbacks, so that there exists a unique (dashed) $\bar h$ such that all the squares commute and are pullbacks. Observe now that, since $h\colon H\to H_1$ is a normal mono, there exists a unique $\chi'\colon H_1\flat H\to H$ such that

$$\begin{tikzcd}
H_1\flat H \arrow[r, "\chi'"] \arrow[d, "1\flat h"'] & H \arrow[d, "h"] \\
H_1\flat H_1 \arrow[r, "\chi"'] & H_1
\end{tikzcd}$$

commutes. From this fact, it follows easily that also

$$\begin{tikzcd}
R\flat(H\times G) \arrow[r, "\chi'\cdot(\bar\sigma\flat\pi_H)"] \arrow[d, "\chi\cdot(1\flat\bar h)"'] & H \arrow[d, "h"] \\
R \arrow[r, "\bar\sigma"'] & H_1
\end{tikzcd}$$

commutes. By the universal property of the pullback of $\overline{\sigma}$ and h, we get a unique morphism x such that

$$
\begin{tikzcd}
R\flat R \arrow[d, "\chi"'] & & R\flat(H\times G) \arrow[ll, "1\flat\overline{h}"'] \arrow[r, "\overline{\sigma}\flat\pi_H"] \arrow[d, "x"'] \arrow[dll, phantom, "(v)"] & H_1\flat H \arrow[d, "\chi'"] \\
R & & H\times G \arrow[ll, "\overline{h}"] \arrow[r, "\pi_H"'] & H
\end{tikzcd}
$$

commutes. The action $\overline{\xi}$ factorizes through x as follows:

$$
\begin{tikzcd}
E\flat(H\times G) \arrow[rr, "\overline{\xi}"] \arrow[dr, "\overline{e}\flat 1"'] & & H\times G \\
& R\flat(H\times G) \arrow[ur, "x"'] &
\end{tikzcd}
$$

To check the commutativity of the previous triangle, compose with the pullback projections

$$
E \xleftarrow{\varphi} H\times G \xrightarrow{\pi_H} H \, .
$$

When composing with φ, use the equality $\varphi = \overline{d}\cdot\overline{h}$ and the left-hand square in the definition of x. When composing with π_H, use the right-hand square in the definition of x and the commutativity of

$$
\begin{tikzcd}
H_0\flat H \arrow[rr, "\xi"] \arrow[dr, "e\flat 1"'] & & H \\
& H_1\flat H \arrow[ur, "\chi'"'] &
\end{tikzcd}
$$

(this last equation is easy to verify: compose with the monomorphism h and use the definition of χ_{H_1}, see diagram (1) of Proposition 2.5).
We are ready to prove the commutativity of diagram (iii): compose with the monomorphism $\iota\colon G\to E$ and, according to our second description of $\overline{\xi}$, replace $\overline{\xi}$ by $x\cdot(\overline{e}\flat 1)$. We get the following commutative diagram

$$
\begin{tikzcd}
E\flat(H\times G) \arrow[rrrr, "1\flat\pi_G"] \arrow[drr, "1\flat\overline{h}"] \arrow[dd, "\overline{e}\flat 1"'] & & & & E\flat G \arrow[ddl, "1\flat\iota"'] \arrow[dd, "\rho\flat 1"] \\
& & E\flat R \arrow[dr, "1\flat\overline{c}"] \arrow[r, phantom, "1\flat(iv)"] & & \\
R\flat(H\times G) \arrow[rr, "1\flat\overline{h}"] \arrow[dd, "x"'] \arrow[dr, phantom, "(v)"] & & R\flat R \arrow[r, "\overline{c}\flat\overline{c}"] \arrow[d, "\chi"'] & E\flat E \arrow[d, "\chi"] & G_0\flat G \arrow[dd, "\xi"] \\
& & R \arrow[r, "\overline{c}"] & E \arrow[d, phantom, "(iv)"] & \\
H\times G \arrow[urr, "\overline{h}"] \arrow[rrrr, "\pi_G"'] & & & & G \arrow[ul, "\iota"']
\end{tikzcd}
$$

where the top-left triangle commutes because $\overline{c} \cdot \overline{e} = 1$, the central square commutes by naturality of χ, and on the right it commutes by condition 3.1.4.

5.3. The universal homomorphism

Combining the equivalence

$$J\colon \mathit{Grpd}(\mathcal{C}) \to \mathit{XMod}(\mathcal{C})$$

of Proposition 2.5 with the construction of the split butterfly E_P associated with a morphism P (see Section 4.1), we are ready to define a homomorphism of bicategories

$$\mathcal{F}\colon \mathit{Grpd}(\mathcal{C}) \to \mathcal{B}(\mathcal{C})\,.$$

On objects and on 1-cells we define

$$\mathcal{F}(\mathbb{H}) = J(\mathbb{H}), \qquad \mathcal{F}(P\colon \mathbb{H} \to \mathbb{G}) = (E_{J(P)}\colon J(\mathbb{H}) \to J(\mathbb{G}))\,.$$

The composition and the identity structural isomorphisms are defined by means of the properties described in Proposition 4.5, by identifying the behavior of $\mathcal{F}$ on 1-cells with the action of the (reduced) composition with the identity butterfly (see Remark 5.5).
It remains to define $\mathcal{F}$ on 2-cells. Let $\alpha\colon P \Rightarrow Q\colon \mathbb{H} \to \mathbb{G}$ be a natural transformation; there exists a unique morphism $\overline{\alpha}$ such that the diagram

$$\begin{tikzcd}
E_{J(P)} \arrow[r, "\sigma_P"] \arrow[dr, "\overline{\alpha}"] \arrow[ddr, bend right, "\overline{p}"'] & H_0 \arrow[dr, "\alpha"] & \\
& G_1 \times_{c,d} G_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & G_1 \arrow[d, "d"] \\
& G_1 \arrow[r, "c"'] & G_0
\end{tikzcd}$$

commutes. Using this morphism, we define $\mathcal{F}(\alpha)\colon E_{J(P)} \to E_{J(Q)}$ as the unique morphism such that the diagram

$$\begin{tikzcd}
E_{J(P)} \arrow[r, "\overline{\alpha}"] \arrow[dr, "\mathcal{F}(\alpha)"] \arrow[ddr, bend right, "\sigma_P"'] & G_1 \times_{c,d} G_1 \arrow[dr, "m"] & \\
& E_{J(Q)} \arrow[r, "\overline{q}"] \arrow[d, "\sigma_Q"'] & G_1 \arrow[d, "c"] \\
& H_0 \arrow[r, "Q_0"'] & G_0
\end{tikzcd}$$

commutes.
Set-theoretically, the map $\mathcal{F}(\alpha)$ sends the pair

$$(y, f\colon x \to p_0(y)) \in E_{J(P)}$$

to the pair

$$(y, \alpha(y) \cdot f\colon x \to q_0(y)) \in E_{J(Q)}.$$

Remark 5.5. Equivalently, $\mathcal{F}\colon \mathit{Grpd}(\mathcal{C}) \to \mathcal{B}(\mathcal{C})$ can be obtained as the composite of $J\colon \mathit{Grpd}(\mathcal{C}) \to XMod(\mathcal{C})$ with the embedding $\mathcal{B}\colon XMod(\mathcal{C}) \to \mathcal{B}(\mathcal{C})$ which is the identity on objects and acts on hom-categories by the reduced composition with the identity butterfly $I_{\mathbb{G}} \cdot_{rc} - : XMod(\mathcal{C})(\mathbb{H}, \mathbb{G}) \to \mathcal{B}(\mathcal{C})(\mathbb{H}, \mathbb{G})$.

Theorem 5.6. *The homomorphism*

$$\mathcal{F}\colon \mathit{Grpd}(\mathcal{C}) \to \mathcal{B}(\mathcal{C})$$

satisfies the universal property of the bicategory of fractions of Grpd($\mathcal{C}$) with respect to the class Σ of weak equivalences.

PROOF. Since the class Σ has a right calculus of fractions (Propositions 5.5 and 5.2 in [48]), we have to prove that $\mathcal{F}$ satisfies conditions BF0 – BF3 of Proposition 5.1.

BF0: Consider a weak equivalence of groupoids and the corresponding morphism $P\colon \mathbb{H} \to \mathbb{G}$ of crossed modules:

$$\begin{tikzcd} H \arrow[r, "p"] \arrow[d, "\partial"'] & G \arrow[d, "\partial"] \\ H_0 \arrow[r, "p_0"'] & G_0 \end{tikzcd}$$

As recalled in Remark 2.6 the arrows induced on kernels and cokernels are isomorphisms. As a first step, we show that the previous diagram is a pullback. For this, consider the regular epi - mono factorizations of the vertical arrows:

$$\begin{tikzcd} H \arrow[r, "p"] \arrow[d, "\partial_1"'] & G \arrow[d, "\partial_1"] \\ I(H) \arrow[r, "I(p)"] \arrow[d, "\partial_2"'] & I(G) \arrow[d, "\partial_2"] \\ H_0 \arrow[r, "p_0"'] & G_0 \end{tikzcd}$$

By Theorem 9.1, the upper square is a pullback because the two regular epimorphisms ∂_1 have isomorphic kernels. As far as the lower square is concerned, observe that $\partial_2\colon I(H) \to H_0$ is normal (precrossed module condition in Section 2.2) and, therefore, it is the kernel of its cokernel. Using this fact, and the fact that the arrow between cokernels is a monomorphism, it is easy to check that the lower square satisfies the universal property of the pullback.
Now, we want to show that the split butterfly $E_P\colon \mathbb{H} \to \mathbb{G}$ associated with the above morphism of crossed modules, as in Section 4.1, is an equivalence. Following Proposition

3.8, it is enough to show that E_P is flippable. For this, consider the diagram

$$\begin{tikzcd}
H \arrow[r, "p"] \arrow[d, "h"'] & G \arrow[dd, "g^\bullet"] \\
H_1 \arrow[d, "\langle d,i\cdot p_1\rangle"'] & \\
E_P \arrow[r, "\overline{p}"] \arrow[d, "\sigma_P"'] & G_1 \arrow[d, "c"] \\
H_0 \arrow[r, "p_0"'] & G_0
\end{tikzcd}$$

The whole rectangle is precisely $P\colon \mathbb{H} \to \mathbb{G}$, so that it is a pullback. The lower square also is a pullback (see Section 4.1), hence the upper square is a pullback. From this and from the fact that $g^\bullet$ is the kernel of $d\colon G_1 \to G_0$, we immediately get that $\langle d, i \cdot p_1\rangle \cdot h$ is a kernel of $d \cdot \overline{p}\colon E_P \to G_0$. Finally, $d \cdot \overline{p}$ is a regular epimorphism by definition of essential surjectivity (see Definition 2.3) and, therefore, it is the cokernel of its kernel.

BF1: Since $\mathcal{F}$ on objects is the composite

$$\mathit{Grpd}(\mathcal{C}) \to \mathit{XMod}(\mathcal{C}) \to \mathcal{B}(\mathcal{C})$$

with the first step being an equivalence and the second one being the identity on objects, condition BF1 is clearly satisfied.

BF2: We are to prove that $\mathcal{F}\colon \mathit{Grpd}(\mathcal{C}) \to \mathcal{B}(\mathcal{C})$ is full and faithful on 2-cells. To this end, let us consider two parallel morphisms of crossed modules $P, Q\colon \mathbb{H} \rightrightarrows \mathbb{G}$ and an arrow $f\colon E_P \to E_Q$ between the corresponding split butterflies (see Section 4.1), i.e. the following four triangles commute:

$$\begin{tikzcd}
H \arrow[r, "\langle \partial, g^\bullet\cdot p\rangle"] \arrow[d, "\langle \partial, g^\bullet\cdot q\rangle"'] \arrow[dr, phantom, "(i)", near start] & E_P \arrow[d, "d\cdot\overline{p}"] \arrow[dl, "f"] \arrow[d, phantom, "(ii)"', near end, shift right=4] \\
E_Q \arrow[r, "d\cdot\overline{q}"'] & G_0
\end{tikzcd}
\qquad
\begin{tikzcd}
E_P \arrow[d, "\sigma_P"'] \arrow[dr, "f"] \arrow[d, phantom, "(iii)", near end, shift left=4] & G \arrow[l, "\langle 0,g\rangle"'] \arrow[d, "\langle 0,g\rangle"] \arrow[dl, phantom, "(iv)", near start] \\
H_0 & E_Q \arrow[l, "\sigma_Q"]
\end{tikzcd}$$

Consider also the arrow π given by the universal property of the pullback E_P:

$$\begin{tikzcd}
H_0 \arrow[r, "e"] \arrow[dr, "\pi"] \arrow[ddr, bend right, "1"'] & H_1 \arrow[dr, bend left, "p_1"] & \\
& E_P \arrow[r, "\overline{p}"] \arrow[d, "\sigma_P"'] & G_1 \arrow[d, "c"] \\
& H_0 \arrow[r, "p_0"'] & G_0
\end{tikzcd}$$

Define

$$\alpha_f \colon\ H_0 \xrightarrow{\ \pi\ } E_P \xrightarrow{\ f\ } E_Q \xrightarrow{\ \overline{q}\ } G_1\ .$$

It is easy to check that $d \cdot \alpha_f = p_0$ and $c \cdot \alpha_f = q_0$: just use commutativity of (ii) and (iii) above. To prove that α is natural requires some computations. Following Definition 2.8, the naturality of α is the same as the commutativity of the diagram

$$\begin{tikzcd}
H \arrow[r, "\partial"] \arrow[d, "\langle p,q\rangle"'] & H_0 \arrow[r, "\pi"] & E_P \arrow[r, "f"] & E_Q \arrow[d, "\overline{q}"] \\
G \times G \arrow[rrr, "m_0"'] & & & G_1
\end{tikzcd}$$

where $m_0 = g \sharp g^{\bullet}$ is the cooperator of g and $g^{\bullet}$. To show the commutativity of this diagram, we present it as the outer rectangle of the diagram

$$\begin{tikzcd}
H \arrow[rr, "\pi\cdot\partial"] \arrow[drr, "\langle 1,p\rangle"'] \arrow[dd, "\langle q,p\rangle"'] & & E_P \arrow[rr, "f"] \arrow[dl, phantom, "(v)"] \arrow[dr, phantom, "(vi)"] & & E_Q \arrow[dd, "\overline{q}"] \\
& & H \times G \arrow[u, "\varphi_P"'] \arrow[urr, "\varphi_Q"'] \arrow[dll, "q\times 1"] \arrow[drr, "(g^{\bullet}\cdot q)\sharp g"] \arrow[rr, phantom, "(vii)"] \arrow[d, phantom, "(viii)"] & & \\
G \times G \arrow[rr, "\tau"'] & & G \times G \arrow[rr, "m_0"'] & & G_1
\end{tikzcd}$$

where the maps φ_P and φ_Q are the cooperators relative to the butterflies E_P and E_Q (see Lemma 5.2), and τ is the twisting isomorphism $G \times G \to G \times G$. The commutativity of (vi), (vii) and $(viii)$ is easily obtained by uniqueness of cooperators, by means of the precompositions with canonical morphisms

$$H \xrightarrow{\langle 1,0\rangle} H \times G \xleftarrow{\langle 0,1\rangle} G\ .$$

Observe that in proving the commutativity of (vi) we use precisely the commutativity of (i) and (iv) above. In fact, $(1, f)$ is precisely the morphism of the spans (determined by the butterflies E_P and E_Q) corresponding to f. Finally we show that (v) commutes, by composing with pullback projections σ_P and $\overline{p}$. Composing with σ_P yields

$$\sigma_P \cdot \pi \cdot \partial = \partial = \partial \cdot \pi_H \cdot \langle 1, p\rangle = \sigma_P \cdot \varphi_P \cdot \langle 1, p\rangle,$$

where the last equality is just the weak equivalence $(\pi_H, \sigma_P) \colon \varphi_P \to \partial$ in the span of crossed modules corresponding to E_P (see Proposition 5.4). Before composing with $\overline{p}$, observe that the following diagram is commutative

$$\begin{tikzcd}
G \arrow[rr, "e\cdot\partial"] \arrow[dr, "\langle 1,1\rangle"'] & & G_1 \\
& G \times G \arrow[ur, "m_0"'] &
\end{tikzcd}$$

This can be easily deduced from the very definition of m_0 as the cooperator of g and $g^{\bullet}$ (this equation is one of the axioms defining a Peiffer graph, see [37]). Hence our computation yields:

$$\overline{p} \cdot \pi \cdot \partial = e \cdot p_0 \cdot \partial = e \cdot \partial \cdot p = m_0 \cdot \langle p, p \rangle = m_0 \cdot \tau \cdot \langle p, p \rangle = m_0 \cdot \tau \cdot (p \times 1) \cdot \langle 1, p \rangle = \overline{p} \cdot \varphi_p \cdot \langle 1, p \rangle \, .$$

The third equality holds by the commutativity of the diagram above, the last one is obtained by observing that each of the morphisms $m_0 \cdot \tau \cdot (p \times 1)$ and $\overline{p} \cdot \varphi_P$ are the cooperator of $g^{\bullet} \cdot p$ and g.

BF3: We want to prove that the diagram in $\mathcal{B}(\mathcal{C})$

$$\begin{tikzcd}
 & {[E]} \arrow[dl, "E_{(\pi_H,\sigma)}"'] \arrow[dr, "E_{(\pi_G,\rho)}"] & \\
\mathbb{H} \arrow[rr, "E"'] & & \mathbb{G}
\end{tikzcd}$$

commutes (up to a 2-cell). On a side we want to construct the butterfly $E_{(\pi_G,\rho)}$. This can be obtained as usual by the reduced composition (see Section 4.2) with the identity $I_{\mathbb{G}} \cdot_{rc} (\pi_G, \rho)$. In fact, we will compute $I_{\mathbb{G}}^{E} \cdot_{rc} (\pi_G, \rho)$, where $I_{\mathbb{G}}^{E}$ is a butterfly (isomorphic to the identity butterfly, see Remark 3.5) more suitable to deal with the pullback projections involved in the proof. We define $I_{\mathbb{G}}^{E} = (\bar{g}^{\bullet}, d, \bar{g}, c)$, where $\bar{g} = i \cdot \bar{\rho} \cdot \langle 0, \iota \rangle$ is a kernel of c with $d \cdot \bar{g} = \partial$, as in the following diagram, where $(i \cdot \bar{\rho}, \rho)$ is a discrete fibration (see Definition 3.3 and the related diagram).

$$\begin{tikzcd}
G \arrow[r, equal] \arrow[d, "\langle 0,\iota \rangle"'] & G \arrow[d, "\bar{g}"] \\
R_\sigma \arrow[r, "i \cdot \bar{\rho}"] \arrow[d, shift right=2, "\sigma_2"'] \arrow[d, shift left=2, "\sigma_1"] & G_1 \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\
E \arrow[r, "\rho"'] \arrow[u, "\Delta"] & G \arrow[u, "e"]
\end{tikzcd}$$

So let us compute $I_{\mathbb{G}}^{E} \cdot_{rc} (\pi_G, \rho) =$

$$\begin{tikzcd}
H \times G \arrow[rr, "\pi_G"] \arrow[dr, "{\langle \varphi, \bar{g}^{\bullet} \cdot \pi_G \rangle}"] \arrow[dd, "\varphi"'] & & G \arrow[dr, "\bar{g}^{\bullet}"] \arrow[dd, "\partial"' near start] & & G \arrow[dl, "\bar{g}"'] \arrow[dd, "\partial"] \\
 & R_\sigma \arrow[rr, "i \cdot \bar{\rho}"] \arrow[dl, "\sigma_1"'] & & G_1 \arrow[dl, "c"] \arrow[dr, "d"'] & \\
E \arrow[rr, "\rho"'] & & G_0 & & G_0
\end{tikzcd}
=
\begin{tikzcd}
H \times G \arrow[dr, "{\langle \varphi, \bar{g}^{\bullet} \cdot \pi_G \rangle}"] \arrow[dd, "\varphi"'] & & G \arrow[dl, "{\langle 0,\iota \rangle}"'] \arrow[dd, "\partial"] \\
 & R_\sigma \arrow[dl, "\sigma_1"] \arrow[dr, "d \cdot i \cdot \bar{\rho}"'] & \\
E & & G_0
\end{tikzcd}$$

where the diagram $\rho \cdot \sigma_1 = c \cdot i \cdot \overline{\rho}$ is a pullback as recalled above.

On the other side we compute $E \cdot_{rc} (\pi_H, \sigma) =$

$$
\begin{tikzcd}
H \times G \arrow[rr, "\pi_H"] \arrow[dr, "\langle \varphi, \kappa\cdot\pi_H \rangle"] \arrow[dd, "\varphi"'] & & H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \\
& R_\sigma \arrow[rr, "\sigma_2"] \arrow[dl, "\sigma_1"] & & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"] & \\
E \arrow[rr, "\sigma"'] & & H_0 & & G_0
\end{tikzcd}
=
\begin{tikzcd}
H \times G \arrow[dr, "\langle \varphi, \kappa\cdot\pi_H \rangle"] \arrow[dd, "\varphi"'] & & G \arrow[dl, "\langle 0, \iota \rangle"'] \arrow[dd, "\partial"] \\
& R_\sigma \arrow[dl, "\sigma_1"] \arrow[dr, "\rho\cdot\sigma_2"] & \\
E & & G_0
\end{tikzcd}
$$

We have to prove that these two butterflies are isomorphic. In fact, we will see that they coincide. This is obvious for the top-right to bottom-left diagonals. Concerning the other diagonals, since $\rho \cdot \sigma_2 = d \cdot i \cdot \bar{\rho}$, we only have to prove the equality $\langle \varphi, \kappa \cdot \pi_H \rangle = \langle \varphi, \bar{g}^\bullet \cdot \pi_G \rangle$, i.e. that the compositions with $i \cdot \bar{\rho}$ are equal. It is sufficient to precompose with $\langle 1_H, 0 \rangle$ and $\langle 0, 1_G \rangle$. The first one gives

$$
i \cdot \bar{\rho} \cdot \langle \varphi, \kappa \cdot \pi_H \rangle \cdot \langle 1_H, 0 \rangle = i \cdot \bar{\rho} \cdot \langle \kappa, \kappa \rangle = i \cdot \bar{\rho} \cdot \Delta_E \cdot \kappa = e \cdot \rho \cdot \kappa = 0
$$

$$
= i \cdot \bar{g} \cdot \pi_G \cdot \langle 1_H, 0 \rangle = i \cdot \bar{\rho} \cdot \langle \varphi, \bar{g}^\bullet \cdot \pi_G \rangle \cdot \langle 1_H, 0 \rangle,
$$

the second one gives

$$
i \cdot \bar{\rho} \cdot \langle \varphi, \kappa \cdot \pi_H \rangle \cdot \langle 0, 1_G \rangle = i \cdot \bar{\rho} \cdot \langle \iota, 0 \rangle = \bar{\rho} \cdot \tau \cdot \langle \iota, 0 \rangle = \bar{\rho} \cdot \langle 0, \iota \rangle
$$

$$
= i \cdot i \cdot \bar{\rho} \cdot \langle 0, \iota \rangle = i \cdot \bar{g} = i \cdot \bar{\rho} \cdot \langle \varphi, \bar{g}^\bullet \cdot \pi_G \rangle \cdot \langle 0, 1_G \rangle,
$$

where τ is the twist isomorphism of R_σ. In both these computations, the first and the last equalities hold by the universal property of the kernel pair R_σ, the other equalities are immediate.

Remark 5.7. The construction given in the proof of condition BF3 above yields a recipe to obtain a crossed module morphism corresponding to a split butterfly. Let us consider the span associated with the split butterfly $E = (\kappa, \iota, \sigma, \rho)$, and suppose we have chosen a section s of the split epimorphism σ. Then, since the left leg of the corresponding span is a pullback diagram, we can pull the section s back along φ, and get the morphism $(\overline{s}, s)$ of crossed modules which, moreover, is a section of (π_H, σ). The situation is summarized in the diagram below:

$$
\begin{tikzcd}
H \arrow[r, shift left, "\overline{s}"] \arrow[d, "\partial"'] & H \times G \arrow[l, shift left, "\pi_H"] \arrow[r, "\pi_G"] \arrow[d, "\varphi"'] & G \arrow[d, "\partial"] \\
H_0 \arrow[r, shift left, "s"] & E \arrow[l, shift left, "\sigma"] \arrow[r, "\rho"'] & G_0
\end{tikzcd}
$$

We can compose the isomorphism $E \cdot E_{(\pi_H, \sigma)} \cong E_{(\pi_G, \rho)}$ with the morphism $(\overline{s}, s)$ and get:

$$
\begin{aligned}
E \;&\cong\; E \cdot_{rc} (\pi_H, \sigma) \cdot (\overline{s}, s) \cong E \cdot E_{(\pi_H, \sigma)} \cdot_{rc} (\overline{s}, s) \cong E_{(\pi_G, \rho)} \cdot_{rc} (\overline{s}, s) \cong \\
&\cong\; I_{\mathbb{G}} \cdot_{rc} (\pi_G, \rho) \cdot (\overline{s}, s) \cong I_{\mathbb{G}} \cdot_{rc} (\pi_G \cdot \overline{s}, \rho \cdot s) \cong E_{(\pi_G \cdot \overline{s}, \rho \cdot s)},
\end{aligned}
$$

i.e. the morphism $(\pi_G \cdot \overline{s}, \rho \cdot s)$ is associated with the (split) butterfly E. Notice the arbitrary choice of the section s: if two different sections are chosen, the associated morphisms of crossed modules are isomorphic.

Remark 5.8. If $\mathcal{C}$ is a category with finite limits (not necessarily semi-abelian), Theorem 5.6 may fail. In this more general case, internal categories may differ from internal groupoids and the bicategory of fractions

$$Cat(\mathcal{C})[\Sigma^{-1}]$$

may still admit a (more involved) explicit description: it is the bicategory of internal anafunctors. This has been proved independently by M. Dupont in [24], where the base category $\mathcal{C}$ is assumed to be regular, and by D. Roberts in [45], where essential surjectivity is intended relatively to a Grothendieck topology on $\mathcal{C}$, and internal categories (not only internal groupoids) are considered.

6. Butterflies and weak morphisms of internal groupoids

From [48], we know that, when $\mathcal{C}$ is the category of groups, then $Grpd(\mathcal{C})[\Sigma^{-1}]$ is the 2-category of groupoids, monoidal functors and monoidal natural transformations (and a similar result holds when $\mathcal{C}$ is the category of Lie algebras). In this section we show explicitly how to construct the "weak morphism" associated with a butterfly in the cases of groups and Lie algebras, and we give an indication of how to recover a butterfly from a weak morphism – indeed the technique can be adapted to other semi-abelian varieties of universal algebra, in order to define a notion of weak morphism.

6.1. The technique

Let us consider a butterfly $E = (E, \kappa, \rho, \iota, \sigma)$ in a semi-abelian variety $\mathcal{C}$ of universal algebra, and let $U : \mathcal{C} \to \mathcal{S}$ be a forgetful functor with $\mathcal{S} = Set_*$ or $\mathcal{S} = k\,Vect$, k being any field. The key properties we are using are that the axiom of choice holds in $\mathcal{S}$, and that the functor U preserves finite limits and sends regular epimorphisms in regular (hence split) epimorphisms.

Let s be a section of $U(\sigma)$ and assume that s preserves 0 (i.e., the unique 0-ary operation, see [17]).

$$\begin{tikzcd} H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \\ & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\ H_0 \arrow[ur, dashed, bend left=10, "s"] & & G_0 \end{tikzcd}$$

We want to show how E yields a weak morphism of groupoids $F_E \colon \mathbb{H} \to \mathbb{G}$.

The functor U preserves finite limits, so that it extends to a 2-functor between the 2-categories of internal groupoids. Now, with the butterfly E, it is associated a span in $Grpd(\mathcal{C})$

$$\mathbb{H} \xleftarrow{S} [E] \xrightarrow{R} \mathbb{G}\ ,$$

with the left leg being a weak equivalence (see Proposition 5.4). More explicitly:

$$\begin{tikzcd}
H_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & (H \times G) \rtimes_{\bar{\xi}} E \arrow[l, "\pi_H \rtimes \sigma"'] \arrow[r, "\pi_G \rtimes \rho"] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & G_1 \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\
H_0 & E \arrow[l, "\sigma"] \arrow[r, "\rho"'] & G_0
\end{tikzcd}$$

By applying U to this construction, S turns into an equivalence in $Grpd(\mathcal{S})$. Actually, S being a (surjective) equivalence means that σ is a regular epimorphism and S is fully faithful, i.e. the following diagram is a pullback:

$$\begin{tikzcd}
(H \times G) \rtimes_{\bar{\xi}} E \arrow[r, "\pi_H \rtimes \sigma"] \arrow[d, "\langle d,c \rangle"'] & H_1 \arrow[d, "\langle d,c \rangle"] \\
E \times E \arrow[r, "\sigma \times \sigma"'] & H_0 \times H_0
\end{tikzcd}$$

Now, since U preserves pullbacks, $U(S)$ is still fully faithful. Moreover, being $U(\sigma)$ a split epimorphism, $U(S)$ is an internal equivalence, whose weak inverse $U(S)^*$ can be computed by pulling back the section $s \times s$ of $U(\sigma \times \sigma) = U(\sigma) \times U(\sigma)$ in the diagram above.
The composition $F_E = U(R) \cdot U(S)^*$ (which is an internal functor in $Grpd(\mathcal{S})$) is a good candidate for a weak morphism in $Grpd(\mathcal{C})$, with the coherence conditions encoded in the butterfly. To simplify notation, in Section 6.2 and in Section 6.3 we assume that G is a subobject of E and $\iota\colon G \to E$ is the inclusion map.

6.2. Case study: groups

Let $\mathcal{C} = Grp$, and $U : Grp \to Set_*$ be the forgetful functor.

Under the equivalence between crossed modules and groupoids, the crossed module $\partial\colon G \to G_0$ gives rise to the groupoid in groups

$$\begin{tikzcd}
G_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & G_0 \arrow[l, "e" description]
\end{tikzcd} ,$$

where G_1 is the semi-direct product $G \rtimes G_0$, with structure maps (additive notation)

$$c\colon (g,x) \mapsto x, \qquad d\colon (g,x) \mapsto \partial g + x, \qquad e\colon x \mapsto (0,x)\,.$$

Following the lines described in the previous section, we get the monoidal functor $F_E = (F_0, F_1, F_2)$, where:

$$\begin{array}{rcll}
F_0 & = & \rho \cdot s\colon H_0 \to G_0; & x \mapsto \rho(s(x)); \\
F_1 & : & H \rtimes H_0 \to G \rtimes G_0; & (h,x) \mapsto (-\kappa(h) + s(\partial(h) + x) - s(x), \rho(s(x))); \\
F_2 & : & H_0 \times H_0 \to G \rtimes G_0; & (x,y) \mapsto (s(x) + s(y) - s(x+y), \rho(s(x+y))).
\end{array}$$

Notice that, since

$$\partial(s(x) + s(y) - s(x+y)) + \rho(s(x+y)) =$$

$$= \rho((s(x)+s(y)-s(x+y))) + \rho(s(x+y)) = \rho(s(x)) + \rho(s(y)),$$

$F_2(x,y)$ is to be interpreted as an arrow $F_2^{x,y}\colon F_0(x)+F_0(y) \to F_0(x+y)$, namely the monoidal structure isomorphism. In fact, $F=(F_0,F_1,F_2)$ is a normalized monoidal functor, i.e. $F_0(0)=0$, because s and ρ preserve 0.

Finally, we give a glance at the construction of a butterfly from a (normalized) monoidal functor of (strict) categorical groups. Consider a functor $F=(F_0,F_1)\colon \mathbb{H} \to \mathbb{G}$ with monoidal structure isomorphisms $F_2^{x_1,x_2}\colon F_0(x_1)+F_0(x_2) \to F_0(x_1+x_2)$. Define P_0 by the following pullback in *Set*

$$\begin{tikzcd} P_0 \arrow[r, "\overline{F}"] \arrow[d, "\sigma"'] & G_1 \arrow[d, "c"] \\ H_0 \arrow[r, "F_0"'] & G_0 \end{tikzcd}$$

and put $\rho = d\cdot\overline{F}\colon P_0 \to G_0$. Despite the fact that F_0 is not a group homomorphism, it can be proved that P_0 is a group and σ and ρ are group homomorphisms (see [48], Proposition 6.3). Moreover, $\ker\sigma = \ker c$ (here we use that $F_0(0)=0$). Finally, given the commutative diagram

$$\begin{tikzcd} H \arrow[r, "F_1\cdot i\cdot h"] \arrow[d, "d\cdot h"'] & G_1 \arrow[d, "c"] \\ H_0 \arrow[r, "F_0"'] & G_0 \end{tikzcd}$$

the canonical arrow κ from H to the pullback P_0 is also a group homomorphism (this follows immediately from the naturality of the monoidal structure of F). This way we get the required butterfly $(P_0,\kappa,\rho,\iota,\sigma)$.

6.3. Case study: Lie algebras

A groupoid in $kLie$, the category of Lie algebras over a fixed field k, is called a strict Lie 2-algebra in [3]. Weak morphisms of Lie 2-algebras are called homomorphisms. Now we consider the forgetful functor $U : kLie \to k\,Vect$. As for the case of groups, it extends to internal groupoids.

Recall that a crossed module in $kLie$ is called a *differential crossed module* (see [3]). This is given by the following data: a morphism of Lie algebras $\partial\colon G \to G_0$ and an (external) action $\cdot\colon G_0 \to \mathrm{Der}(G)$, such that, for any $g_1, g_2, g \in G$ and $x \in G_0$,

$$\begin{aligned} \partial(x\cdot g) &= [x,\partial(g)] \qquad \text{(precrossed module condition)},\\ \partial(g_1)\cdot g_2 &= [g_1,g_2] \qquad \text{(Peiffer identity)}. \end{aligned}$$

Thanks to the equivalence of both internal actions and external actions with the category of points in $kLie$, the definition of crossed module given above is equivalent to the internal one given in [31]. We describe explicitly the construction of the groupoid associated with the crossed module ∂:

$$G_1 = G \rtimes G_0 \;\underset{c}{\overset{d}{\rightrightarrows}}\; G_0 \;\text{ (with } e\colon G_0 \to G_1\text{)},$$

where the semi-direct product $G \rtimes G_0$ is the vector space $U(G) \oplus U(G_0)$ endowed with the bracket operation defined by

$$[(g_1, x_1), (g_2, x_2)] = ([g_1, g_2] + x_1 \cdot g_1 - g_2 \cdot x_2, [x_1, x_2]),$$

and the structural maps of the groupoid are defined:

$$c((g, x)) = x, \qquad d((g, x)) = \partial(g) + x, \qquad e(x) = (0, x).$$

With notation as above, we can define F_E with the technique described in Section 6.1. Indeed the construction of F_0 and F_1 only uses the additive structure of Lie algebras, so that they are defined precisely as in the case of groups. On the other side, F_2 involves the bracket operations, and, applying the same technique as in the case of groups, one obtains

$$F_2 \colon (x, y) \mapsto ([s(x), s(y)] - s([x, y]), \rho(s[x, y]))\,.$$

7. Classification of extensions

In this section we assume that $\mathcal{C}$ has split extension classifiers (see [9, 11], and Section 9), as it happens, for instance, in the category of groups (where the split extension classifier $[G]$ associated with an object G is the group $\mathrm{Aut}(G)$ of automorphisms of G) or of Lie algebras (where $[G]$ is the Lie algebra $\mathrm{Der}(G)$ of derivations of G).

Consider two objects H and G in $\mathcal{C}$. Let $D(H) = (0 \to H)$ be the discrete crossed module on H and

$$\mathcal{A}(G) = (\mathcal{I}_G \colon G \to [G]\,,\ \mathrm{ev} \colon [G]\flat G \to G)$$

the crossed module associated with $[G]$ (that is, the crossed module corresponding to the action groupoid, see [14]). The following lemma generalizes Example 13.4 of [40].

Lemma 7.1. *The groupoid*

$$\mathit{Ext}(H, G)$$

of extensions of the form $G \to E \to H$ is isomorphic to the groupoid

$$\mathcal{B}(\mathcal{C})(D(H), \mathcal{A}(G))\,.$$

Such an isomorphism restricts to split extensions and split butterflies.

Proof. Let us start with a butterfly $D(H) \xrightarrow{E} \mathcal{A}(G)$:

$$\begin{tikzcd}
0 \arrow[dd, "!"'] \arrow[dr, "!"] & & G \arrow[dl, "\iota"'] \arrow[dd, "\mathcal{I}_G"] \\
& E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\
H & & {[G]}
\end{tikzcd}$$

We are going to prove that ρ is uniquely determined. Since ι is normal in E, there exists a unique action χ' such that

$$\begin{tikzcd} E\flat G \arrow[r, "\chi'"] \arrow[d, "1\flat\iota"'] & G \arrow[d, "\iota"] \\ E\flat E \arrow[r, "\chi_E"'] & E \end{tikzcd}$$

commutes. Following Remark 3.2, the right wing of the butterfly determines a discrete fibration of groupoids. Hence the diagram

$$\begin{tikzcd} G \rtimes_{\chi'} E \arrow[r, shift left, "p_E"] \arrow[d, "1\rtimes\rho"'] \arrow[dr, phantom, "(i)"] & E \arrow[l, shift left] \arrow[d, "\rho"] \\ G \rtimes_{\mathrm{ev}} [G] \arrow[r, shift left, "p_{[G]}"] & [G] \arrow[l, shift left] \end{tikzcd}$$

is a pullback, i.e. $\chi' = \mathrm{ev} \cdot (\rho\flat 1)$, and ρ is univocally determined by the universal property of $[G]$.

Conversely, consider a short exact sequence

$$G \xrightarrow{\iota} E \xrightarrow{\sigma} H \ .$$

By the universal property of $[G]$, we get a unique ρ such that diagram (i) above is a pullback, so that $\chi' = \mathrm{ev} \cdot (\rho\flat 1)$. It remains to show that the short exact sequence (ι, σ), equipped with ρ, is a butterfly from $D(H)$ to $\mathcal{A}(G)$. Condition 3.1.4 follows by the commutativity of the following diagram:

$$\begin{tikzcd} E\flat G \arrow[r, "1\flat\iota"] \arrow[d, "\rho\flat 1"'] \arrow[dr, "\chi'"] & E\flat E \arrow[dr, "\chi_E"] & \\ {[G]}\flat G \arrow[r, "\mathrm{ev}"'] & G \arrow[r, "\iota"'] & E \end{tikzcd}$$

while Condition 3.1.3 is trivial. As far as the commutativity of the right wing is concerned, since both $\rho\cdot\iota$ and $\mathcal{I}_G$ classify the same split extension, by the universal property we conclude that they are equal.

Combining the previous isomorphism of groupoids with Theorem 5.6, we get a very general classification of extensions:

$$\mathit{Ext}(H,G) \simeq \mathcal{B}(\mathcal{C})(D(H), \mathcal{A}(G)) \simeq \mathit{Grpd}(\mathcal{C})[\Sigma^{-1}](D(H), \mathcal{A}(G))\,.$$

Putting together this classification and the results stated in Section 6, we can conclude that:

1. Group extensions with kernel G and cokernel H are classified by monoidal functors from $D(H)$ to $\mathcal{A}(G)$;
2. Lie algebra extensions with kernel G and cokernel H are classified by homomorphisms of Lie 2-algebras from $D(H)$ to $\mathcal{A}(G)$.

Example 7.2. From the classical cohomological classification of group extensions (see Section 8 of Chapter 4 in [36], for instance) we know that with an extension

$$G \xrightarrow{\kappa} E \xrightarrow{\sigma} H$$

and a chosen set-theoretical section s of σ, we can associate two set-theoretical functions

$$F_0 \colon H \to \mathrm{Aut}(G)\,, \qquad F_0(x)(g) = s(x) + g - s(x)\,,$$

$$f \colon H \times H \to G\,, \qquad f(x,y) = s(x) + s(y) - s(x+y)\,,$$

such that for any x, y, z in H the equations

$$F_0(x)(f(y,z)) + f(x, y+z) = f(x,y) + f(x+y, z)\,, \tag{3}$$

$$F_0(x) \cdot F_0(y) = \mathcal{I}_G(f(x,y)) \cdot F_0(x+y)\,, \tag{4}$$

hold, where, for an element g of G, $\mathcal{I}_G(g)$ is the corresponding inner automorphism of G. Now we can define two set-theoretical functions F_1 and F_2 as follows:

$$F_1 \colon H \to G \rtimes \mathrm{Aut}(G) \qquad F_1(x) = (0, F_0(x))\,,$$

$$F_2 \colon H \times H \to G \rtimes \mathrm{Aut}(G) \qquad F_2(x,y) = (f(x,y), F_0(x+y))\,.$$

The functions F_0, F_1 and F_2 are a special case of those described in Section 6.2 and form a monoidal functor

$$F_E = (F_0, F_1, F_2) \colon D(H) \to \mathcal{A}(G)\,.$$

Indeed, condition (3) expresses the coherence of the monoidal structure isomorphism with the associativity of $D(H)$ and $\mathcal{A}(G)$, and condition (4) expresses the fact that the pair (F_0, F_1) commutes with the domain maps of $D(H)$ and $\mathcal{A}(G)$.
Let us insist on the fact that the map F_0 is not in general a homomorphism and $\mathcal{I}_G \cdot f$ measures precisely how much F_0 deviates from being a homomorphism. The map F_0 is actually a homomorphism when the section s is a homomorphism (this is the case of split extensions), or when the kernel G is abelian. In this last case, equation (3) is nothing but a cocycle condition, while equation (4) amounts to the fact that F_0 is a homomorphism.

8. The free exact case

When $\mathcal{C}$ is the category of groups, the main result of [40] is not stated in terms of bicategory of fractions, but it is stated as an equivalence of groupoids

$$\mathcal{B}(\mathcal{C})(\mathbb{H}, \mathbb{G}) \simeq \mathit{XMod}(\mathcal{C})(\mathbb{K}, \mathbb{G})\,,$$

where $\mathbb{K}$ is the crossed module of groups obtained from $\mathbb{H}$ by pulling back $\partial \colon H \to H_0$ along a surjective homomorphism $K_0 \to H_0$, with K_0 being a free group. The same is done for Lie algebras in [1]. The aim of this section is to generalize the previous equivalence to the case when the semi-abelian category $\mathcal{C}$ is also free exact.

Recall from [22] that $\mathcal{C}$ is free exact if it has enough regular projective objects. This means that, for every object X in $\mathcal{C}$, there exists a regular epimorphism $x\colon X' \to X$ with X' regular projective. All semi-abelian varieties of universal algebra are of this kind. In particular, groups and Lie algebras are free exact semi-abelian categories.

Let $\mathbb{C}$ be a groupoid and $s_0\colon X_0 \to C_0$ be a regular epimorphism, with X_0 regular projective. Consider the limit

$$\begin{tikzcd} & & X_1 \arrow[dll, "d"'] \arrow[d, "s_1"] \arrow[drr, "c"] & & \\ X_0 \arrow[dr, "s_0"'] & & C_1 \arrow[dl, "d"] \arrow[dr, "c"'] & & X_0 \arrow[dl, "s_0"] \\ & C_0 & & C_0 & \end{tikzcd}$$

The internal graph $d, c\colon X_1 \rightrightarrows X_0$ inherits a structure of groupoid from that of $\mathbb{C}$. Moreover, the internal functor $S = (s_1, s_0)\colon \mathbb{X} \to \mathbb{C}$ is a weak equivalence (it is full and faithful by construction of s_1, and it is essentially surjective, because s_0 is a regular epimorphism). Finally, observe that, since X_0 is regular projective, the groupoid $\mathbb{X}$ is Σ-projective: every weak equivalence with codomain $\mathbb{X}$ is, in fact, an equivalence. We call $S\colon \mathbb{X} \to \mathbb{C}$ a Σ-projective replacement of $\mathbb{C}$.

Proposition 8.1. *Let $\mathbb{C}$ and $\mathbb{D}$ be groupoids and fix a Σ-projective replacement $S\colon \mathbb{X} \to \mathbb{C}$. There is an equivalence of groupoids*

$$\mathcal{B}(\mathcal{C})(J(\mathbb{C}), J(\mathbb{D})) \simeq \mathit{Grpd}(\mathcal{C})(\mathbb{X}, \mathbb{D})\,.$$

Proof. Since $S\colon \mathbb{X} \to \mathbb{C}$ is a weak equivalence, $\mathcal{F}(S)\colon J(\mathbb{X}) \to J(\mathbb{C})$ is an equivalence (see condition BF0 in the proof of Theorem 5.6). Therefore, $\mathcal{F}(s)$ induces an equivalence

$$\mathcal{B}(\mathcal{C})(J(\mathbb{C}), J(\mathbb{D})) \simeq \mathcal{B}(\mathcal{C})(J(\mathbb{X}), J(\mathbb{D}))\,.$$

Moreover, since X_0 is regular projective, all extensions of the form $X_0 \leftarrow E \leftarrow D$ split, and then all butterflies from $J(\mathbb{X})$ to $J(\mathbb{D})$ split:

$$\mathcal{B}(\mathcal{C})(J(\mathbb{X}), J(\mathbb{D})) = \mathcal{B}(\mathcal{C})(J(\mathbb{X}), J(\mathbb{D}))_{\mathrm{split}}\,.$$

Finally, following Section 4.1, we have

$$\mathcal{B}(\mathcal{C})(J(\mathbb{X}), J(\mathbb{D}))_{\mathrm{split}} \simeq \mathit{Grpd}(\mathcal{C})(\mathbb{X}, \mathbb{D})\,.$$

Remark 8.2. To end this section, we sketch a general argument on bicategories of fractions which subsumes Proposition 8.1.
Let Σ be a class of 1-cells in a bicategory $\mathcal{B}$ with a right calculus of fractions. Assume that:

1. Σ satisfies the $2 \Rightarrow 3$ property: let $F\colon \mathbb{C} \to \mathbb{D}$ and $G\colon \mathbb{D} \to \mathbb{E}$ be 1-cells in $\mathcal{B}$; if two of F, G and $G \cdot F$ are in Σ, then the third one is in Σ.

2. Every 1-cell $W\colon \mathbb{C} \to \mathbb{D}$ of Σ is full and faithful, that is, for any object $\mathbb{A}$, the functor $\mathcal{B}(\mathbb{A}, W)\colon \mathcal{B}(\mathbb{A}, \mathbb{C}) \to \mathcal{B}(\mathbb{A}, \mathbb{D})$ is full and faithful.
3. For every object $\mathbb{C}$ in $\mathcal{B}$, there exists $S\colon \mathbb{X} \to \mathbb{C}$ in Σ, with $\mathbb{X}$ a Σ-projective object.

Then, if we fix objects $\mathbb{C}$ and $\mathbb{D}$ and a 1-cell $S\colon \mathbb{X} \to \mathbb{C}$ as in 3, the functor assigning to a 1-cell $F\colon \mathbb{X} \to \mathbb{D}$ the span

$$\begin{tikzcd} & \mathbb{X} \arrow[dl, "S"'] \arrow[dr, "F"] & \\ \mathbb{C} & & \mathbb{D} \end{tikzcd}$$

yields an equivalence of groupoids

$$\mathcal{B}(\mathbb{X}, \mathbb{D}) \simeq \mathcal{B}[\Sigma^{-1}](\mathbb{C}, \mathbb{D})\,.$$

9. A reminder on semi-abelian categories

The general context where the theory of internal crossed modules and weak maps takes place, is that of semi-abelian categories with an additional assumption: equivalence relations centralize each other in the sense of Smith if and only if the corresponding normal monomorphisms commute in the sense of Huq. In the following, the basic notions are recalled and the notation is fixed, for the reader's convenience.

9.1. Protomodular and semi-abelian categories

Semi-abelian categories were introduced in 2002 [33], and they represent the state-of-the-art in the long-lasting investigations whose aim is to provide an abstract categorical setting for non (necessarily) commutative pointed algebraic structures, such as groups, rings or Lie algebras.

A category is semi-abelian when it is pointed (i.e. $0 = 1$), with finite coproducts, protomodular [12] and exact (in the sense of Barr).
Pointed protomodular categories can be characterized as pointed, finitely complete categories where the split short five lemma holds: given a diagram

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "h"'] & A \arrow[r, shift left, "p"] \arrow[d, "f"] & B \arrow[l, shift left, "s"] \arrow[d, "g"] \\ K' \arrow[r, "k'"'] & A' \arrow[r, shift left, "p'"] & B' \arrow[l, shift left, "s'"] \end{tikzcd}$$

where k and k' are kernels of p and p' respectively, $f \cdot k = k' \cdot h$, $g \cdot p = p' \cdot f$ and $f \cdot s = s' \cdot g$, the morphism f is an isomorphism if h and g are.
Recall that a category is exact (in the sense of Barr [4]) when it is regular, and internal equivalence relations are effective, i.e. kernel pairs. Finally a regular category is a finitely complete category where effective equivalence relations have pullback stable coequalizers.

The protomodularity condition can be reformulated when the category $\mathcal{C}$ is pointed regular. This is stated in the following useful characterization (see Theorem 2.3 in [16] or Theorem 4.2 in [19]).

Theorem 9.1. *Let $\mathcal{C}$ be a regular pointed category. Then $\mathcal{C}$ is protomodular if and only if the following property holds: in any commutative diagram*

$$\begin{tikzcd}
\operatorname{Ker} p \arrow[r, "k"] \arrow[d, "h"'] & A \arrow[r, "p"] \arrow[d, "f"'] \arrow[dr, phantom, "(i)"] & B \arrow[d, "g"] \\
\operatorname{Ker} p' \arrow[r, "k'"'] & A' \arrow[r, "p'"'] & B'
\end{tikzcd}$$

if p is a regular epi, then h is an iso if and only if the square (i) is a pullback.

9.2. The "Huq = Smith" condition

In order to introduce the so-called "Huq = Smith" condition, we first recall the notions of commuting subobjects and of centralizing each other equivalence relations.

Two subobjects

$$G \xrightarrow{g} E \xleftarrow{h} H$$

commute in the sense of Huq (see [30, 18]) if they cooperate as morphisms, i.e. if there exists a (unique) morphism φ (called the cooperator of g and h) such that the diagram

$$\begin{tikzcd}
G \arrow[r, "\langle 1,0\rangle"] \arrow[dr, "g"'] & G \times H \arrow[d, dashed, "\varphi"] & H \arrow[l, "\langle 0,1\rangle"'] \arrow[dl, "h"] \\
& E &
\end{tikzcd}$$

commutes.

Suppose that the maps g and h are normal monomorphisms, i.e. kernels. Then the denormalized version of the above notion is that of centralizing each other equivalence relations. A pair of equivalence relations on a common object E

$$\begin{tikzcd}
R \arrow[r, shift left=2, "r_0"] \arrow[r, shift right=2, "r_1"'] & E \arrow[l, "e_R" description] \arrow[r, "e_S" description] & S \arrow[l, shift right=2, "s_0"'] \arrow[l, shift left=2, "s_1"]
\end{tikzcd}$$

centralize each other (in the sense of Smith, see [46, 43]) when there exists a (unique) morphism Φ such that the diagram

$$\begin{tikzcd}
R \arrow[r, "\langle 1,e_S\cdot r_1\rangle"] \arrow[dr, "r_0"'] & R \times_{r_1,s_0} S \arrow[d, dashed, "\Phi"] & S \arrow[l, "\langle e_R\cdot s_0,1\rangle"'] \arrow[dl, "s_1"] \\
& E &
\end{tikzcd}$$

commutes.

It is a well known fact that, when two equivalence relations centralize each other, then their normalizations commute (see [18]). The converse does not hold in general, not even in semi-abelian categories (see [13] for a counterexample, due to G. Janelidze, in the semi-abelian category of digroups). Nevertheless it does hold in several important algebraic contexts, as, for instance, pointed strongly protomodular categories (see [18], Section 6) and pointed action accessible categories (see [37]). As a matter of fact, for internal structures in (many) pointed varieties of universal algebra, this is quite a crucial notion and it recaptures the feeling that a local behavior near the identity element determines a global behavior. Furthermore, it has been acknowledged in [32] that this property is a candidate to become an axiom for "good" semi-abelian categories. We will refer to it as to the "Huq = Smith" property.

Remark 9.2. Two morphisms cooperate if their images do, and this happens precisely when their commutator is trivial, for a suitable notion of commutator. The categorical version of Smith's commutator has been introduced by M.C. Pedicchio in [43]. Unfortunately, the description of the several aspects of the commutator theory involved would take us far beyond our purposes. The interested reader may refer to [37], and the bibliography therein.

9.3. Butterflies in a Grothendieck topos

Butterflies were originally defined by B. Noohi for crossed modules of groups [40], but the author himself, in [2] with E. Aldrovandi, extends the construction to crossed modules of internal groups in any Grothendieck topos $\widehat{\mathsf{S}}$ of sheaves over a site (S, J), with subcanonical topology J.

The present setting generalizes the one of [2]. In fact more is true: our results apply to any pointed strongly protomodular algebraic theory in a Grothendieck topos. To see this, it is necessary to recollect some results from the literature.
First, in [10], Example 4.6.3 shows that if $\mathbb{T}$ is a pointed protomodular algebraic theory, and $\mathcal{C}$ is a regular (exact) category, then the category $\mathrm{Alg}_{\mathbb{T}}(\mathcal{C})$ of models of $\mathbb{T}$ in $\mathcal{C}$ is homological (exact homological). Hence, if $\mathcal{C}$ is exact, the missing condition for $\mathrm{Alg}_{\mathbb{T}}(\mathcal{C})$ to be semi-abelian is its finite cocompleteness.
Indeed, the category of models of an algebraic theory in an elementary topos $\mathcal{E}$ is finitely cocomplete, if (i) the topos has a Natural Number Object, and (ii) the theory is finitely presented. Back to the situation considered here, for a Grothendieck topos $\mathcal{E}$, condition (i) holds, and condition (ii) can be dropped (see [10] again, the discussion after the cited example), so that $\mathrm{Alg}_{\mathbb{T}}(\mathcal{E})$ is semi-abelian.
Concerning the condition "Huq = Smith", strongly protomodular semi-abelian (i.e. strongly semi-abelian) categories have this property, and we know from [8] that for a strongly protomodular (not necessarily pointed) theory $\mathbb{T}$, and a finitely complete category $\mathcal{E}$, the category of models $\mathrm{Alg}_{\mathbb{T}}(\mathcal{E})$ is still strongly protomodular. This is clearly the case for a Grothendieck topos $\mathcal{E}$.

In conclusion, we can state that not only our constructions and results apply to the situation described in [2], but also in the context of internal Lie algebras, internal rings and other strongly semi-abelian theories defined in a Grothendieck topos $\mathcal{E}$.

9.4. Internal object actions

Several notions of actions exist in many algebraic contexts. Most of them share the disadvantage of not being defined intrinsically, but by means of set-theoretical maps satisfying certain properties. From an algebraic-categorical point of view, this is not convenient, since those maps are difficult to deal with. This issue has been fixed by the notion of internal action [16, 9], that expresses its full classifying power in the context of semi-abelian categories.

Let $\mathcal{C}$ be a finitely complete pointed category with coproducts. Then, for any object B in $\mathcal{C}$, one can define a functor "Ker" from the category of split epimorphisms (points) over B into $\mathcal{C}$:

$$\mathrm{Ker}\,: Pt_B(\mathcal{C}) \to \mathcal{C}, \qquad \begin{tikzcd} A \arrow[d, shift left, "p"] \\ B \arrow[u, shift left, "s"] \end{tikzcd} \mapsto \mathrm{Ker}\,p\,.$$

This has a left adjoint:

$$\mathrm{B}+(-): \mathcal{C} \to Pt_B(\mathcal{C}), \qquad X \mapsto \begin{tikzcd} B+X \arrow[d, shift left, "{[1,0]}"] \\ B \arrow[u, shift left, "i_B"] \end{tikzcd}\,.$$

The monad corresponding to this adjunction is denoted by $B\flat(-)\colon \mathcal{C} \to \mathcal{C}$, and, for any object A of $\mathcal{C}$, we obtain a kernel diagram:

$$B\flat A \xrightarrow{j_{B,A}} B+A \xrightarrow{[1,0]} B\,.$$

The $B\flat(-)$-algebras are called internal B-actions in $\mathcal{C}$. In the case of groups, the object $B\flat A$ is the group generated by the formal conjugates of elements of A by elements of B, i.e. by the triples of the kind (b, a, b^{-1}), with $b \in B$ and $a \in A$.

For any object A of $\mathcal{C}$, one can define a canonical conjugation action of A on A itself given by the composition:

$$\chi_A:\ A\flat A \xrightarrow{j_{A,A}} A+A \xrightarrow{[1,1]} A\,.$$

In the category of groups, the morphism χ_A is the internal action associated with the usual conjugation in A: the realization morphism $[1,1]$ of above makes the formal conjugates of $A\flat A$ computed effectively in A.

Finally, observe that conjugation actions are components of a natural transformation $\chi\colon (-)\flat(-) \Rightarrow \mathrm{Id}_{\mathcal{C}}$.

[1] O. Abbad, Categorical classifications of extensions, Ph. D. Thesis.

[2] E. Aldrovandi and B. Noohi, Butterflies I: Morphisms of 2-group stacks, *Advances in Mathematics* **221** (2009) 687–773.

[3] J.C. Baez and A.S. Crans, Higher-dimensional algebra VI: Lie 2-algebras, *Theory and Applications of Categories* **12** (2004) 492–528.

[4] M. Barr, Exact categories, *Springer LNM* **236** (1971) 1–120.

[5] J. Bénabou, Introduction to bicategories, *Springer LNM* **40** (1967) 1–77.

[6] J. Bénabou, Les distributeurs, *Université catholique de Louvain, Institut de Mathématique Pure et Appliquée, rapport 33* (1973).

[7] F. Borceux, Handbook of Categorical Algebra 1, *Cambridge University Press* (1994).

[8] F. Borceux, Non-pointed strongly protomodular theories, *Applied Categorical Structures* **12** (2004) 319–338.
[9] F. Borceux, G. Janelidze and G.M. Kelly, Internal object actions, *Commentationes Mathematicae Universitatis Carolinae* **46** (2005) 235–255.
[10] F. Borceux and D. Bourn, Mal'cev, Protomodular, Homological and Semi-abelian Categories, *Kluwer Academic Publishers* (2004).
[11] F. Borceux and D. Bourn, Split extension classifier and centrality, *Contemp. Math.,* **43** (2007) 85-104.
[12] D. Bourn, Normalization equivalence, kernel equivalence and affine categories, *Springer LNM* **1488** (1991) 43–62.
[13] D. Bourn, Commutator theory in strongly protomodular categories, *Theory and Applications of Categories* **13** (2004) 27–40.
[14] D. Bourn, Action groupoid in protomodular categories, *Theory and Applications of Categories* **16** (2006) 46–58.
[15] D. Bourn, Internal profunctors and commutator theory; applications to extensions classification and categorical Galois Theory, *Theory and Applications of Categories* **24** (2010) 451–488.
[16] D. Bourn and G. Janelidze, Protomodularity, descent, and semidirect products, *Theory and Applications of Categories* **4** (1998) 37–46.
[17] D. Bourn and G. Janelidze, Characterization of protomodular varieties of universal algebras, *Theory and Applications of Categories* **11** (2003) 143–147.
[18] D. Bourn and M. Gran, Centrality and normality in protomodular categories, *Theory and Applications of Categories* **9** (2001) 151–165.
[19] D. Bourn and M. Gran, Regular, protomodular and abelian categories. In: Categorical Foundations, M.C. Pedicchio and W. Tholen Editors, *Cambridge University Press* (2004) 165–211.
[20] M. Bunge and R. Paré, Stacks and equivalence of indexed categories, *Cahiers de Topologie et Géométrie Différentielle Catégorique* **20** (1979) 373–399.
[21] A. Carboni, M.C. Pedicchio and N. Pirovano, Internal graphs and internal groupoids in Mal'cev categories, *CMS Conference Proceedings* **13** (1992) 97–109.
[22] A. Carboni and E.M. Vitale, Regular and exact completions, *Journal of Pure and Applied Algebra* **125** (1998) 79–116.
[23] P. Deligne, La formule de dualité globale, *Springer LNM* **305** (1973) 481-587.
[24] M. Dupont, Abelian metamorphosis of anafunctors into butterflies, preprint (2008).
[25] J. Elgueta, On the regular representation of an (essentially) finite 2-group, *Advances in Mathematics* **227** (2011) 170–209.
[26] T. Everaert, R.W. Kieboom and T. Van der Linden, Model structures for homotopy of internal categories, *Theory and Applications of Categories* **15** (2005) 66–94.
[27] A. Fröhlich and C.T.C. Wall, Graded monoidal categories, *Compositio Mathematica* **28** (1974) 229–285.
[28] P. Gabriel and M. Zisman, Calculus of Fractions and Homotopy Theory, *Springer, Berlin* (1967).
[29] D.F. Holt, An interpretation of the cohomology groups $H^n(G, M)$, *Journal of Algebra* **60** (1979) 307–318.
[30] S.A. Huq, Commutator, nilpotency, and solvability in categories, *Quarterly Journal of Mathematics Oxford Second Series* **19** (1968) 363–389.
[31] G. Janelidze, Internal crossed modules, *Georgian Mathematical Journal* **10** (2003) 99–114.
[32] G. Janelidze, Advances in semi-abelian categorical algebra, *talk at Category Theory Meeting, Genova* (2010).
[33] G. Janelidze, L. Márki and W. Tholen, Semi-abelian categories, *Journal of Pure and Applied Algebra* **168** (2002) 367–386.
[34] G. Janelidze, L. Márki and A. Ursini, Ideals and clots in universal algebra and in semi-abelian categories, *Journal of Algebra* **307** (2007) 191–208.
[35] P.T. Johnstone, Topos theory, *Academic Press, London* (1977).

[36] S. Mac Lane, Homology, *Springer, Berlin* (1963).

[37] S. Mantovani and G. Metere, Internal crossed modules and Peiffer condition, *Theory and Applications of Categories* **23** (2010) 113–135.

[38] S. Mantovani, G. Metere and E.M. Vitale, Profunctors in Malt'sev categories and fractions of functors, *Journal of Pure and Applied Algebra*, (2012) in press, `DOI: 10.1016/j.jpaa.2012.10.015`.

[39] N. Martins-Ferreira and T. Van der Linden, A note on the "Smith is Huq" condition, *Applied Categorical Structures* **20** (2012) 175–187.

[40] B. Noohi, On weak maps between 2-groups, `arXiv:math/0506313v3` (2005) .

[41] B. Noohi, Notes on 2-groupoids, 2-groups and crossed modules, *Homology, Homotopy and Applications* **9** (2007) 75–106.

[42] B. Noohi, Integrating morphisms of Lie 2-algebras, `arXiv:0910.1818v3` (2009).

[43] M.C. Pedicchio, A categorical approach to commutator theory, *Journal of Algebra* **177** (1995) 647–657.

[44] D. Pronk, Etendues and stacks as bicategories of fractions, *Compositio Mathematica* **102** (1996) 243–303.

[45] D. Roberts, Internal categories, anafunctors and localisations, `arXiv:1101.2363v1` (2011).

[46] J.D.H. Smith, Mal'cev varieties, Springer LNM **554** (1976).

[47] E.M. Vitale, A Picard-Brauer exact sequence of categorical groups, *Journal of Pure and Applied Algebra* **175** (2002) 383–408.

[48] E.M. Vitale, Bipullbacks and calculus of fractions, *Cahiers de Topologie et Géométrie Différentielle Catégorique* **51** (2010) 83–113.