# Some fibrational properties of normal monomorphisms

Giuseppe Metere

*Università degli Studi di Palermo*

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# Overview

# 1. There are several ways of being normal

# There are several ways of being normal

The study of congruences in terms of their zero-classes is well established in algebra, and it has determined the introduction of different kinds of substructures.

In the first part of my talk, I will recall some classes of monomorphisms (subobjects) arising this way.

The ambient category $\mathcal{C}$ will be assumed to be pointed regular, i.e.

- $0 \cong 1$
- $\mathcal{C}$ is finitely complete
- kernel pairs have coequalizers
- regular epis are pullback stable

# Kernels

A map $k : K \to A$ is a kernel when there exists a map $f : A \to B$ such that the following diagram is a pullback:

$$\begin{array}{ccc} K & \longrightarrow & 0 \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow \\ A & \underset{f}{\longrightarrow} & B \end{array}$$

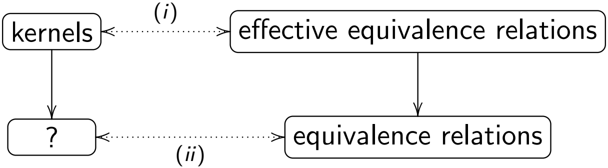
In a pointed variety, the kernel of a homomorphism $f : A \to B$ is the inverse image along $f$ of the constant $0 \in B$.

# Kernels

The kernel of a morphism $f$ is the zero-class of the kernel-pair (= effective) equivalence relation determined by $f$.

If $\mathcal{C}$ is Barr exact (all equiv. relations are effective), kernels control the equivalence relations in $\mathcal{C}$.

If $\mathcal{C}$ is normal (pt.d regular with regular = normal epi's), kernels are still zero-classes of effective equivalence relations (= *kernel pairs* of their cokernels), but they cannot describe the collection of all the equivalence relations in $\mathcal{C}$.



# Normal monomorphisms

In 2000, Bourn introduces a notion of normal monomorphism as a categorical interpretation of a class of an equivalence relation in $\mathcal{C}$. When $\mathcal{C}$ is pointed, such a notion has an easier description: a normal mono is just the zero-class of an equivalence relation:

$$k \text{ is normal to } R \quad \Longleftrightarrow \quad \begin{array}{ccc} K & \longrightarrow & R \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow{\scriptstyle \langle r_1, r_2 \rangle} \\ A & \xrightarrow[\langle 0,1 \rangle]{} & A \times A \end{array}$$

**Fact** ([Bourn, 2000]). When $\mathcal{C}$ is pointed and protomodular (i.e. split short-five-lemma holds), there is a 1:1 correspondence between normal subobjects and equivalence relations (up to isos).

# Ideals ans Clots

Recall the notions of **ideals** ([Higgins 1956] and [Magari 1967]) and **clots** ([Aglianò and Ursini, 1992]) for a pointed variety $\mathcal{C}$.

(a) A term $t(\vec{x}, \vec{y})$ is an ideal term in $\vec{y}$ if $t(\vec{x}, \vec{0}) = 0$ is an identity of $\mathcal{C}$.

(b) A subalgebra $K$ of $A$ in $\mathcal{C}$ is an ideal in $A$ if

$$\vec{k} \in K^n \Rightarrow t(\vec{a}, \vec{k}) \in K$$

$\forall\ \vec{a} \in A^m$ and $\forall$ ideal term $t$.

(c) A subalgebra $K$ of $A$ in $\mathcal{C}$ is a clot in $A$ if

$$t(\vec{a}, \vec{0}) = 0 \text{ and } \vec{k} \in K^n \ \Rightarrow t(\vec{a}, \vec{k}) \in K$$

$\forall\ \vec{a}, \vec{k} \in A^m \times A^n$ and $\forall$ $(m+n)$-ary term $t$ of $A$.

# Ideals ans Clots ...after [G. Janelidze, Márki and Ursini, 2009]

In a pt.d category $\mathcal{C}$, with kernels and fin. coproducts, a subobject $k\colon K \to A$ is a clot if it is closed under conjugation in $A$.

$$\begin{array}{ccc} A\flat K & \dashrightarrow & K \\ \downarrow & & \downarrow \\ A\flat A & \xrightarrow[\text{conj.}]{} & A \end{array}$$

If $\mathcal{C}$ is regular, $K$ is a clot iff it is the 0-class (0-star!) of a *reflective relation*.

In a pt.d regular category $\mathcal{C}$ with finite coproducts, a subobject $k\colon K \to A$ is an ideal if it the regular image of a clot.

$$\begin{array}{ccc} H & \xrightarrow{f'} \!\!\!\!\rightarrow & K \\ {\scriptstyle \text{clot}}\downarrow & & \downarrow{\scriptstyle k} \\ B & \xrightarrow[f]{} \!\!\!\!\rightarrow & A \end{array}$$

## Normalities in p.ted reg. cats. w. finite coprod. [Mantovani, M., 2010]

| KERNEL | NORMAL | CLOT | IDEAL |
|---|---|---|---|
| $\begin{array}{ccc} K & \longrightarrow & 0 \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow \\ A & \xrightarrow[f]{} & B \end{array}$ | $\begin{array}{ccc} K & \longrightarrow & R \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow \\ A & \xrightarrow{\langle 0,1\rangle} & A\times A \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_1} \\ 0 & \longrightarrow & A \end{array}$ $K=[0]_R$, $R$ equ.rel. | $\begin{array}{ccc} K & \longrightarrow & R \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow \\ A & \xrightarrow{\langle 0,1\rangle} & A\times A \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_1} \\ 0 & \longrightarrow & A \end{array}$ $K=[0]_R$, $R$ refl.rel. | $\begin{array}{ccc} H & \overset{f'}{\twoheadrightarrow} & K \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ B & \underset{f}{\twoheadrightarrow} & A \end{array}$ $H$ clot, $f, f'$ reg.epi |

KERNEL $\Longrightarrow$ NORMAL $\Longrightarrow$ CLOT $\Longrightarrow$ IDEAL

NORMAL $\Longleftarrow$ IDEAL (Mal'tsev)

# [Mantovani, M., 2010] + [Martins-Ferreira, Montoli, Ursini, Van der Linden 2017]

| KERNEL | NORMAL | CLOT | IDEAL |
|---|---|---|---|
| $\begin{array}{ccc} K & \longrightarrow & 0 \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow \\ A & \xrightarrow[f]{} & B \end{array}$ | $\begin{array}{ccc} K & \longrightarrow & R \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow \\ A & \xrightarrow{\langle 0,1\rangle} & A \times A \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_1} \\ 0 & \longrightarrow & A \end{array}$ $K = [0]_R$, $R$ equ.rel. | $\begin{array}{ccc} K & \longrightarrow & R \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow \\ A & \xrightarrow{\langle 0,1\rangle} & A \times A \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_1} \\ 0 & \longrightarrow & A \end{array}$ $K = [0]_R$, $R$ refl.rel. | $\begin{array}{ccc} K & \longrightarrow & R \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow \\ A & \xrightarrow{\langle 0,1\rangle} & B \times A \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_1} \\ 0 & \longrightarrow & B \end{array}$ $K = [0]_R$, $R$ surj. l.split rel. |

KERNEL $\Longrightarrow$ NORMAL $\Longrightarrow$ CLOT $\Longrightarrow$ IDEAL

NORMAL $\Longleftarrow$ IDEAL: Mal'tsev

**Problem.** Kernels determine their associated effective equivalence relation. What about normal monos?

**Definition.** A morphism $n$ is normalizing if *there exists* an equivalence relation $R$ such that $n$ is normal to $R$,...

...unfortunately, there is no known general construction of this equivalence relation, given a subobject 'candidate to be normal'. [Borceux, 2004]

One aim of [M., 2017] was to find such a general construction.

Do not necessarily assume $\mathcal{C}$ pointed

$\Rightarrow$ need the complete defn. of normal subobject.

Do not expect to get a bijection normalizing m.$\longleftrightarrow$equiv. relations

$\Rightarrow$ get a functor **Rel**, with $n$ normal to $\mathbf{Rel}(n)$.

**Definition.** [Bourn, 2000] In a category $\mathcal{C}$ with finite limits, a morphism $n\colon N \to X$ is *normal* to an equivalence relation $(R, r_1, r_2)$ on the object $X$ when $n \times n$ factors via $\langle r_1, r_2\rangle$ and the following two commutative diagrams are pullbacks:

$$
\begin{array}{ccc}
N \times N & \xrightarrow{\tilde{n}} & R \\
\| & \lrcorner & \downarrow{\scriptstyle \langle r_1, r_2\rangle} \\
N \times N & \xrightarrow[n\times n]{} & X \times X
\end{array}
\qquad\qquad
\begin{array}{ccc}
N \times N & \xrightarrow{\tilde{n}} & R \\
{\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle r_1} \\
N & \xrightarrow[n]{} & X
\end{array}
$$

$$
\textit{i.e. } n^{-1}(R) = \nabla_N \qquad \textit{i.e. } (\tilde{n}, n)\colon \nabla_N \to R \text{ disc. fibr.}
$$

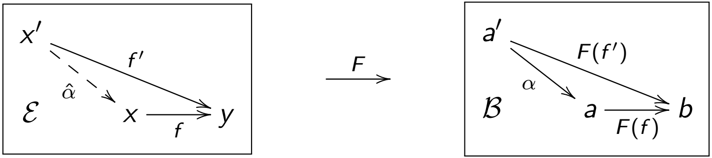
suggest to investigate the opfibrational aspects of the fibration:

$$
(\ )_0\colon \mathbf{EqRel}(\mathcal{C}) \to \mathcal{C} \qquad\qquad \begin{array}{c} R \\ \downarrow\downarrow \\ X \end{array} \mapsto X
$$

# 2. Bifibrational point of view

# Bifibrations

A morphism $f\colon x \to y$ is cartesian w.r.t. the functor $F\colon \mathcal{E} \to \mathcal{B}$, if, for all $\alpha\colon a' \to a$ and $f'\colon x' \to y$ with $F(f') = F(f)\cdot\alpha$, there is a unique lifting $\hat{\alpha}\colon x' \to x$ with $F(\hat{\alpha}) = \alpha$ and $f' = f\cdot\hat{\alpha}$.




$F$ is a fibration if there are enough cartesian liftings.
$F$ is an opfibration if $F^{op}$ is a fibration, i.e. there are enough opcartesian liftings.
$F$ is a bifibration if it is both a fibration and an opfibration.

## Why bifibrations? (spoiler alert!)

**Proposition.** When $(\ )_0 \colon \mathbf{EqRel}(\mathcal{C}) \to \mathcal{C}$ is a bifibration, then a mono $N \xrightarrow{n} X$ is normalizing iff its opcartesian lifting

$$\nabla_N \longrightarrow n_!(\nabla_N)$$

is a disc. fib. in $\mathbf{EqRel}(\mathcal{C})$; e.g. $n$ is normal to $\mathbf{Rel}(n) := n_!(\nabla_N)$.

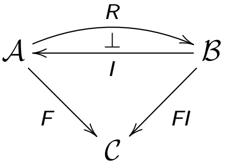
**Proposition.** [M., 2017] If $\mathcal{C}$ is a regular Mal'tsev category with pushouts of split monos, then $(\ )_0$ is a bifibration.

**Proposition.** [Bourn, 2018] If $\mathcal{C}$ is finitely complete with infima of equivalence relations, then $(\ )_0$ is a bifibration.

In the rest of my talk, I'd like to show how the result in [M., 2017] can be deduced from a more general setting, and how the same setting could be used to deal with ideals and clots.

# Restriction of a bifibration to a reflective subcategory

**Proposition.** [M., 2017] Consider the diagram




where $(R \dashv I, \eta, \epsilon)$ is a full replete epi-reflection, $F$ is a bifibration and $\eta$ has $F$-vertical components. Then the adjunction restricts to fibres, and the following hold:

- every $F$-cartesian map with its codomain in $\mathcal{B}$ is itself in $\mathcal{B}$;
- $FI$ is a bifibration with same cartesian liftings as $F$ and opcartesian liftings obtained by reflecting $F$-opcartesian ones;
- $I$ is a cartesian functor over $\mathcal{C}$

# Left split spans and relations

As a main example, we introduce the reflection of left split spans into left split relations.

**Definition.** Let $\mathcal{C}$ be a category with finite limits. The category $\mathbf{sSpan}(\mathcal{C})$ has objects *left split spans*, i.e. diagrams in $\mathcal{C}$:

$$X \underset{e}{\overset{d}{\leftrightarrows}} Z \xrightarrow{c} Y \qquad \text{s.t. } de = 1$$

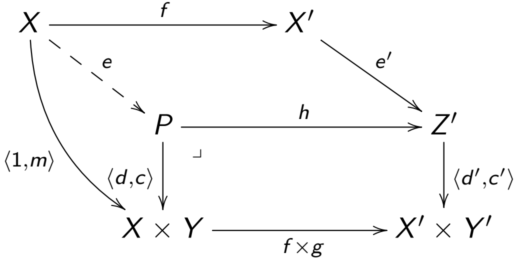
Morphisms of left split spans are morphisms of diagrams.

A functor $U\colon \mathbf{sSpan}(\mathcal{C}) \to \mathbf{Arr}(\mathcal{C})$ is defined

$$U\colon (d, e, c) \mapsto c \cdot e$$

**Proposition.** $U\colon \mathbf{sSpan}(\mathcal{C}) \to \mathbf{Arr}(\mathcal{C})$ is a fibration.

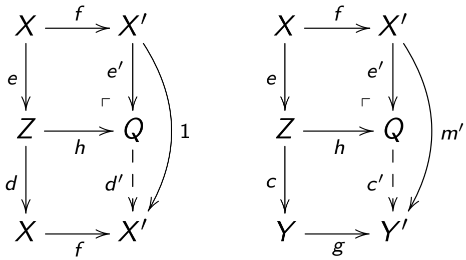
*Idea:*




$(f, h, g)\colon (d, e, c) \to (d', e', c')$ is the cart. lifting of $(f, g)$ at $Z'$.

**Proposition.** If $\mathcal{C}$ has pushouts of split monos, then the functor $U\colon \mathbf{sSpan}(\mathcal{C}) \to \mathbf{Arr}(\mathcal{C})$ is an opfibration.

*Idea:*




$(f, h, g)\colon (d, e, c) \to (d', e', c')$ is the opcart. lifting of $(f, g)$ at $Z$.

**Definition.** The category $\mathbf{sRel}(\mathcal{C})$ of **left split relations** in $\mathcal{C}$ is the full subcategory of $\mathbf{sSpan}(\mathcal{C})$ determined by those spans

$$X \underset{e}{\overset{d}{\leftrightarrows}} S \xrightarrow{c} Y$$

such that $d$ and $c$ are jointly monic.

**Proposition.** If the finitely complete category $\mathcal{C}$ is regular, the full replete canonical inclusion

$$I\colon \mathbf{sRel}(\mathcal{C}) \to \mathbf{sSpan}(\mathcal{C})$$

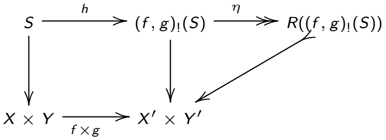
admits a left adjoint that makes $\mathbf{sRel}(\mathcal{C})$ a regular epi-reflective subcategory of $\mathbf{sSpan}(\mathcal{C})$.

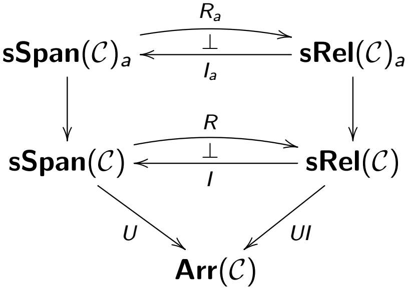
*Idea:* take the regular epimorphic image of the span.

When $\mathcal{C}$ is a regular category with pushouts of split monos, we can apply our general result on the restriction of a bifibration.

Then $UI$ is a bifibration

- with same cart. lifting, as $U$
- with opcart. liftings:



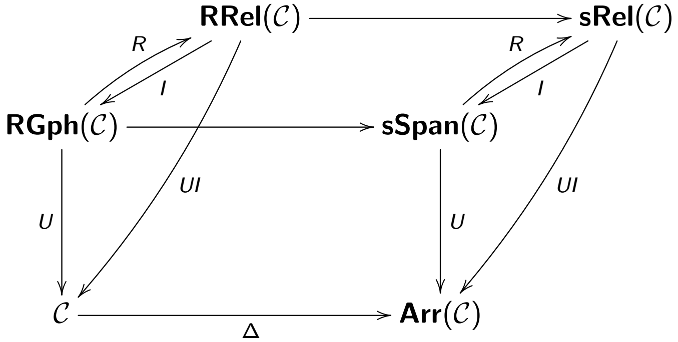




- $R \dashv I$ restricts to fibres

# 3. Back to normality

# The case of reflexive relations

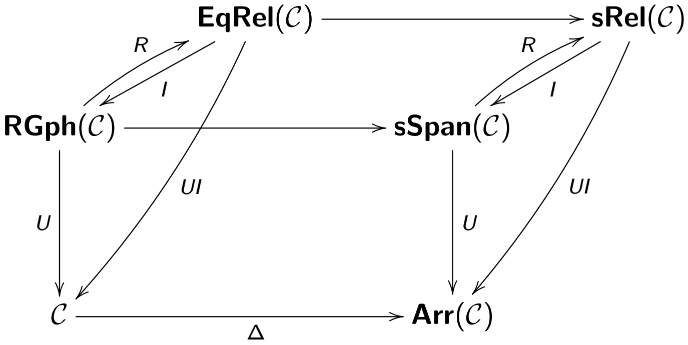
Let $\mathcal{C}$ regular with pushouts of split monos. Take the pullback of $R \dashv I$ along the diagonal $\Delta \colon \mathcal{C} \to \mathbf{Arr}(\mathcal{C})$ sending $X$ to $1_X$:




In this case, our general setting can be used to study the reflection of reflexive graph into reflexive relations.

# The case of equivalence relations

Let $\mathcal{C}$ Mal'tsev regular with pushouts of split monos. Take the pullback of $R \dashv I$ along the diagonal $\Delta\colon \mathcal{C} \to \mathbf{Arr}(\mathcal{C})$.




Since in Mal'tsev, reflexive relations coincide with equivalence relations.

**Proposition.** [M., 2017] Let $\mathcal{C}$ be a regular Mal'tsev category, with pushouts of split monos. Given an arrow $N \xrightarrow{n} X$ of $\mathcal{C}$, the assignment

$$\mathbf{Rel}(n) = R(n_!(\nabla N))$$

defines a functor

$$\mathbf{Rel}\colon \mathbf{Arr}(\mathcal{C}) \longrightarrow \mathbf{EqRel}(\mathcal{C})$$

Its restriction to normal monos

$$\mathbf{Rel}\colon \mathbf{Nor}(\mathcal{C}) \longrightarrow \mathbf{EqRel}(\mathcal{C})$$

is faithful, and $n$ is normal to $\mathbf{Rel}(n)$.

**Proposition.** [M., 2017] $\mathbf{Rel}(n)$ is the initial equivalence relation among the equivalence relations to which $n$ is normal.

The following characterization removes the existential quantifier from the definition of normalizing monomorphism.

**Corollary.** [M., 2017] Let $\mathcal{C}$ be a Mal'tsev regular category, with pushouts of split monos. An arrow $N \xrightarrow{n} X$ is normalizing if and only if its opcartesian lifting

$$\nabla N \xrightarrow{\hat{n}} R(n_!(\nabla N))$$

is a (cartesian) discrete fibration in **EqRel**($\mathcal{C}$).

**Example.** If $\mathcal{C}$ is a pointed variety of Universal Algebra, given a subalgebra $(N, n)$ of an algebra $X$, **Rel**($n$) is the subalgebra of $X \times X$ generated by $\Delta_X(X) \cup (N \times N)$, where $\Delta_X(X)$ is the diagonal of $X \times X$.

**Rel**($n$) is a semi-congruence (a congruence when $\mathcal{C}$ is Mal'tsev).

**Theorem.** [M., 2017] Let $\mathcal{C}$ be a pointed regular Mal'tsev category with p.o. of split monos. There is a mono coreflection:

$$\mathbf{Nor}(\mathcal{C}) \underset{\mathbf{Rel}}{\overset{\mathbf{Nor}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} \mathbf{EqRel}(\mathcal{C})$$

where $\mathbf{Nor}\colon \mathbf{Nor}(\mathcal{C}) \to \mathbf{EqRel}(\mathcal{C})$ is the normalization functor. When $\mathcal{C}$ is also protomodular, the last is an adjoint equivalence, that restricts to the fibres. Taking classes (=subobjects), one gets an isomorphism of posets:

$$[\mathbf{Nor}]_X(\mathcal{C}) \underset{\mathbf{Rel}}{\overset{\mathbf{Nor}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\cong}}}} [\mathbf{EqRel}]_X(\mathcal{C})$$

# 4. An ideal end

# An ideal end

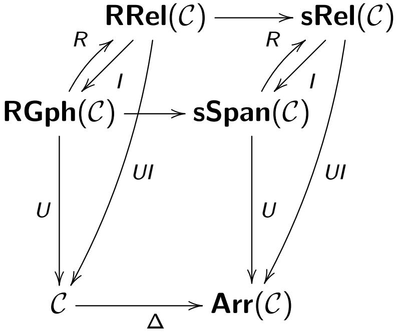
The machinery of the bifibration of left split spans developed in the second part of the talk seems to be oversized for dealing with normal monos in regular Malt'sev categories.

In the last few slides I will discuss some other possible applications, which are all work in progress.

# Clots

For $\mathcal{C}$ regular with p.o. of split monos, the fibred adjunction can be used to study the reflection of reflexive graphs into reflexive relations.

When $\mathcal{C}$ is also pointed, this seems a promising setting to investigate the properties of clots, which are 0-classes of semi-congruences.



# Left punctual relations

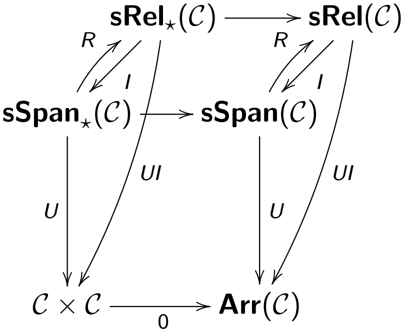
For $\mathcal{C}$ pointed regular with p.o. of split monos, consider the functor

$$0\colon \mathcal{C} \times \mathcal{C} \to \mathbf{Arr}(\mathcal{C})$$

given by extending

$$(X, Y) \mapsto 0_{X,Y}$$

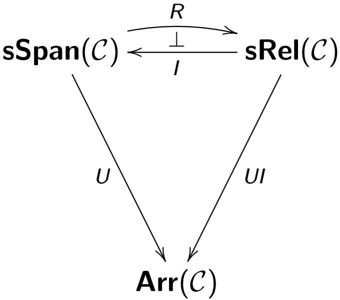
Here the fibred adjunction gives the reflection of left punctual spans into left punctual relations.



# Ideals

For $\mathcal{C}$ regular with p.o. of split monos, the original fibred adjunction can be used to study the reflection of surjective left split spans into surjective left split relations.

When $\mathcal{C}$ is also pointed, this seems a promising setting to investigate the properties of ideals, which are 0-classes surjective left split relations.



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