Weak inverses for strict n-categories

Stefano Kasangian, Giuseppe Metere, Enrico M. Vitale

March 23, 2009

Abstract

In this paper we show that the two apparently different notions of weak inverses for a strict n-category due to R. Street and to Kapranov and Voevodsky do coincide[1].

1 Introduction

Topological templates of higher categorical structures suggest to consider the definition of invertible cells since the beginning of the developing of the theory. This is straightforward when considering strict inverses for strict n-categories, but in a weak n-categorical setting, we need week–inverses to be defined.
A gentle approach to this issue is considering a semi–strict intermediate case: namely that of weak inverses for strict n-categories. The idea is that the general case would reduce to this when specialized to strict n-categories.
In literature there are (at least) two such.
The first goes back to the classics on this subject: Streets' paper on *The algebra of oriented simplexes* [Str87]. Here, a (weakly) invertible k-cell is one that has a weak-inverse with respect to $(k-1)$-composition. Specializing this definition for 0-inverses, one can say that an arrow $f : x \to y$ is weakly invertible whenever there is an arrow $g : y \to x$, such that $f \circ g$ is equivalent to 1_x and $g \circ f$ is equivalent to 1_y. The first assertion means that there is an invertible 2-cell $fg \Rightarrow 1_x$; hence there is an inverse 2-cell going in the opposite direction, similarly for the second assertion. Hence by induction, in a strict n-category we have to produce (at least) 2^n witnesses in order to say that an arrow is weakly invertible. Let us notice that all those witnesses are required to obey no coherence law at all.
A second approach is supplied by M. M. Kapranov and V. A. Voevodsky (CT meeting in Bangor, 1989, published in [KV91]MR1130401) where a notion of invertible cell is closely connected with that of equivalence of n-categories. Their definition in dimension n is based upon the idea that a n-groupoid is a strict n-category where all equations of the form $ax = b$ (and $xa = b$) admit a weak solution. Here a, x and b are cells of any (possibly different) dimension and the *multiplication* is any composition. Of course composites are meant to be composeable.
Kapranov and Voevodsky conjectured that the their notion of inverse cell implies that of Street, but the latter do not imply the former. As an argument to this

[1]Financial support by INDAM ????, FNRS grant ???? and Dote Ricercatore (Regione Lombardia) are gratefully acknowledged.

conjecture, they show how their definition implies the existence for any invertible cell of a *coherent system of quasi inverses*, i.e. a higher dimensional analog of triangular identities, while "...there is no way to construct such a system from Street's definition" [KV91].
We disprove this conjecture by showing that the two notions are indeed equivalent. The proof is obtained by double induction, over the dimension and the level of localization, and uses the idea that a n-functor is an equivalence when it is h-surjective at any level of localization, including the top degenerate one, that of the n^{th} localization where surjective (on equations) means injective on hom-sets.
The paper is organized as follows: next section is devoted to present the sesquicategory n-Cat of strict n-categories, strict n-functors and weak n-transformations; in the following one we give a notion of weak equivalence of n-categories. The precise relation between the notion of weakly invertible and Street–invertible cells is summarized in Theorem 4.2, which is stated in section four, where the 2-dimensional case is studied in detail. Finally the last section occupies almost half of the paper with the proof of the main result.
The results contained in the present work have been presented at the International Category Theory Conference in Calais, June 2008. A similar result has been announced by Y. Lafont at HoCat 2008 Conference in Barcelona (see [LMW07]).

2 Preliminaries

In this section we recall first a standard construction of the category n-Cat, of (small) strict–n-categories and their morphisms, inductively enriched over the category $(n-1)$-Cat (see for instance [Str87]). A new perspective is given by considering the richer structure of *sesqui–category* of n-Cat, necessary in order to take into account the 2-morphisms, namely lax–n-transformations, and their compositions. Details can be found in [Met08].

Sesqui–categories where defined by Street [Str96]. The term *sesqui* comes from the latin *semis-que*, that means (one and) a half. Hence a sesqui-category is something in-between a category and a 2-category. More precisely, a sesqui-category $\mathcal{C}$ is a category $\lfloor\mathcal{C}\rfloor$ with a lifting of the hom-functor to Cat, such that the following diagram of categories and functors commutes:

$$\begin{array}{ccc} & & \text{Cat} \\ & \nearrow^{\mathcal{C}(-,-)} & \big\downarrow \scriptstyle{\mathbf{obj}} \\ \lfloor\mathcal{C}\rfloor^{\text{op}} \times \lfloor\mathcal{C}\rfloor & \xrightarrow[\lfloor\mathcal{C}\rfloor(-,-)]{} & \mathbf{Set} \end{array}$$

Objects and morphisms of $\lfloor\mathcal{C}\rfloor$ are also objects and 1-cells of $\mathcal{C}$, while morphisms of $\mathcal{C}(A,B)$'s (with A and B running in $\mathbf{obj}(\lfloor\mathcal{C}\rfloor)$) are the 2-cells of $\mathcal{C}$.

2.1 The category n-Cat

For $n = 0, 1$ we can safely consider the usual category of small sets and categories respectively. Hence let us suppose $n > 1$.

A (strict) n-category $\mathbb{C}$ consists of a set of objects $\mathbb{C}_0$, and for every pair $c_0, c_0' \in \mathbb{C}_0$, a $(n-1)$-category $\mathbb{C}_1(c_0, c_0')$. Structure is given by morphisms of $(n-1)$-categories:

$$\mathbb{I} \xrightarrow{u^0(c_0)} \mathbb{C}_1(c_0, c_0) \ , \qquad \mathbb{C}_1(c_0, c_0') \times \mathbb{C}_1(c_0', c_0'') \xrightarrow{\circ^0_{c_0, c_0', c_0''}} \mathbb{C}_1(c_0, c_0'') \ ,$$

called resp. 0-units and 0-compositions, with c_0, c_0', c_0'' any triple of objects $\mathbb{C}_0$. Axioms are the usual for strict unit and strict associativity.

Notation: Cell dimension will be often recalled as subscript, as c_k is a k-cell in the n-category $\mathbb{C}$. Moreover, if

$$c_k : c_{k-1} \to c'_{k-1} : c_{k-2} \to c'_{k-2} : \cdots \to \cdots : c_1 \to c_1' : c_0 \to c_0',$$

c_k can be considered as an object of the $(n-k)$-category

$$\Big[\cdots \big[[\mathbb{C}_1(c_0, c_0')]_1(c_1, c_1') \big]_1 \cdots \Big]_1 (c_{k-1}, c'_{k-1}).$$

In order to avoid this quite uncomfortable notation, the latter will be renamed more simply $\mathbb{C}_k(c_{k-1}, c'_{k-1})$, while with $\mathbb{C}_k$ we will mean the disjoint union of all such. Finally, 0-subscript of the underlying set of an n-category, will be often omitted.

A morphism of n-categories is a (strict) n-functor $\mathbb{F} : \mathbb{C} \to \mathbb{D}$. It consists of set-theoretical map $F_0 : \mathbb{C}_0 \to \mathbb{D}_0$ together with morphisms of $(n-1)$-categories

$$F_1^{c_0, c_0'} : \mathbb{C}_1(c_0, c_0') \longrightarrow \mathbb{D}_1(F_0 c_0, F_0 c_0')$$

for any pair of objects c_0, c_0' of $\mathbb{C}_0$, such that usual (strict) functoriality axioms are satisfied. Let us notice that subscripts and superscripts will be often omitted, when this does not cause confusion.
Routine calculations shows that these data organizes in a category.

2.2 The sesqui–categorical structure of n-Cat

The category Set can be easily endowed with a trivial sesqui–categorical structure. For $n = 1$, the category Cat is a 2-category, with natural transformations as 2-cells. Hence it has an underlying sesqui–category.
Again we can suppose $n > 1$. For given n-functors $F, G : \mathbb{C} \to \mathbb{D}$, a lax n-transformation $\alpha : F \to G$ consists of a pair (α_0, α_1) where the first is a map $\alpha_0 : \mathbb{C}_0 \to \mathbb{D}_1$ such that $\alpha_0(c_0) = \alpha_{c_0} : Fc_0 \to Gc_0$, and $\alpha_1 = \{\alpha_1^{c_0, c_0'}\}_{c_0, c_0' \in \mathbb{C}_0}$ is a collection of 2-morphisms of $(n-1)$-categories

$$\begin{array}{ccccc} & & \mathbb{C}_1(c_0, c_0') & & \\ & \swarrow^{F_1^{c_0,c_0'}} & & \searrow^{G_1^{c_0,c_0'}} & \\ \mathbb{D}_1(F_0c_0, F_0c_0') & & \xLeftarrow{\alpha_1^{c_0,c_0'}} & & \mathbb{D}_1(G_0c_0, G_0c_0') \\ & \searrow_{-\circ^0 \alpha_0 c_0'} & & \swarrow_{\alpha_0 c_0 \circ^0 -} & \\ & & \mathbb{D}_1(F_0c_0, G_0c_0') & & \end{array} \qquad (1)$$

satisfying suitable functoriality axioms [Met08]. In the sequel we will refer to diagrams like (1) as to *naturality diagrams* for the 2-morphism α. A n-transformation is called strict when all $\alpha_1^{-,-}$ are identities.
It is possible to define in a natural way a vertical composition of 2-morphisms, and left and right compositions with a 1-morphisms (whiskering). It is not difficult to show that these data define a sesqui–category [Met08].

3 Equivalences of n-Categories and weak inverses

In this section we give a notion of weak equivalence suitable for morphisms of n-categories.

Definition 3.1. *Let a n-category morphism $F : \mathbb{C} \to \mathbb{D}$ be given.*
For $n = 0$, a weak equivalence of 0-categories is a bijective map.
Hence let us suppose $n > 0$. F is a weak equivalence of n-categories *if it satisfies conditions* (1) *and* (2) *below:*

1. *for every object d_0 in $\mathbb{D}$, there exists an object c_0 in $\mathbb{C}$ and a 1-cell $d_1 : d_0 \to Fc_0$ such that for every d_0' in $\mathbb{C}$, the morphism*
$$d_1 \circ - : \mathbb{D}_1(Fc_0, d_0') \to \mathbb{D}_1(d_0, d_0')$$
is a weak equivalence of $(n-1)$-categories;
2. *for every pair c_0, c_0' in $\mathbb{C}$, $F_1^{c_0,c_0'} : \mathbb{C}_1(c_0, c_0') \to \mathbb{D}_1(Fc_0, Fc_0')$ is a weak equivalence of $(n-1)$-categories.*

According to Definition 3.1, we define equivalence–cells:

Definition 3.2. *A 1-cell $c_1 : c_0 \to c_0'$ of a n-category $\mathbb{C}$ is* weakly invertible*, when for every object c_0'' of $\mathbb{C}$, the morphism*
$$c_1 \circ - : \mathbb{C}_1(c_0', c_0'') \to \mathbb{C}_1(c_0, c_0'')$$
is a weak equivalence of $(n-1)$-categories.

Let us notice that a consequence of Definition 3.2 is a notion of weak equivalence for cells of every dimension. In fact, a k-cell $c_k : c_{k-1} \to c_{k-1}'$ of $\mathbb{C}$ is a weak equivalence if it is so when considered as a 1-cell of $\mathbb{C}_k(c_{k-1}, c_{k-1}')$.
Remark 3.3. It is interesting to translate our notion of weak equivalences into the globular point of view. In order to unfold the induction in Definition 3.1, let us examine it in low dimension.
Let $n = 1$, then $F : \mathbb{C} \to \mathbb{D}$ is just a functor between categories. Condition (1) means F is h-surjective. In fact such $d_1 : d_0 \to Fc_0$ provided by the definition is an isomorphic cell, as composition with it induces 0–equivalences (i.e. isomorphisms) on hom-sets. Clearly condition (2) means F is full and faithful, so that what we get is exactly a categorical equivalence. Carrying on a similar analysis for $n = 2$ would take us to the well known notion of bi–equivalence, and so on.
A careful look at these examples leads to the conclusion that a morphism of n-categories $F : \mathbb{C} \to \mathbb{D}$ is a weak equivalence if each of its localizations is h-surjective, means surjective up to a weakly invertible cell. In order to make this precise we must reformulate the case $n = 0$ in more general terms: a set-theoretical map $F : \mathbb{C} \to \mathbb{D}$ is a 0-equivalence if and only if

1. it is h-surjective on object, that is, surjective up to equations, that is, surjective,
2. F is surjective on equations, that is, for every $c, c' \in \mathbb{C}$, $Fc = Fc'$ implies $c = c'$, that is, F is injective.

Hence injectivity is a kind of degenerate surjectivity! Let us point out that these conditions are exactly those of Definition 3.1 if we let $\mathbb{C}_{n+1}(c_n, c'_n)$ be the singleton if $c_n = c'_n$, the empty set otherwise.

Finally we can state a globular formulation for equivalences of general n: a morphism of n-categories $F : \mathbb{C} \to \mathbb{D}$ is a weak equivalence if for every pair of k-cells c_k, c'_k $(-1 \leq k \leq n)$ with same $(k-1)$-domain and $(k-1)$-codomain, the localization

$$F^{c_k, c'_k}_{k+1} : \mathbb{C}_{k+1}(c_k, c'_k) \longrightarrow \mathbb{D}_{k+1}(Fc_k, Fc'_k)$$

is h-surjective.

The cases when $k < 1$ deserve an explanation. For $k = 0$ we should think of our n-categories $\mathbb{C}$ and $\mathbb{D}$ as suspended over the distinguished objects $\star_1$ and $\star_2$, i.e. as $(n+1)$-categories with only 2 objects (with their identities). The 1-cells $\star_1 \to \star_2$ in the suspension are labeled by the objects of $\mathbb{C}$ and $\mathbb{D}$ resp., the 2-cells are the 1-cells of the n-categories and so on. Of course the n-functor F extends trivially to a $(n+1)$-functor which preserves the two distinguished base points. The case $k = -1$ is dealt similarly, it yields the usual notion of h-surjectivity on objects.

Remark 3.4. Simpson has given in [Sim98] a notion of weakly invertible cell, that is apparently stronger than that of Definition 3.2. Actually he shows that his definition is equivalent to the corresponding notion given in [KV91]. It will be a Corollary to our Main Theorem the fact that our Definition 3.2 is equivalent to that of Simpson. This will fix the apparent asymmetry of our definitions.

4 Invertible cells and inverses

Purpose of this section is to relate Definition 3.2 with a notion of equivalence based on the existence of weak-inverses. This has been introduced for ω-categories by Street in [Str87]. For the reader's convenience, we recall his definition in the setting considered here.

Definition 4.1 (Street). *Two objects c_0, c'_0 of an n-category $\mathbb{C}$ are equivalent if there exist 1-cells $c_1 : c_0 \to c'_0$ and $c^*_1 : c'_0 \to c_0$ such that*

- *$c_1 \circ c^*_1$ and 1_{c_0} are equivalent in $\mathbb{C}_1(c_0, c_0)$*
- *$c^*_1 \circ c_1$ and $1_{c'_0}$ are equivalent in $\mathbb{C}_1(c'_0, c'_0)$.*

We will call an arrow *Street–invertible* if it establishes an equivalence (according to Street) between its domain and its co-domain.

Street definition inductively induces systems of inverses for a given Street–invertible cell. It is useful to spell it out in low dimensions.

Following Definition 4.1, two elements of a set are equivalent precisely when they are equal. Hence for a category $\mathbb{C}$, an arrow c_1 is Street–invertible if there exists a c_1^* such that $1_{c_0} = c_1 \circ^0 c_1^*$ and $c_1^* \circ^0 c_1 = 1_{c_0'}$, that is, if c_1 is an isomorphism In a 2-category $\mathbb{C}$, a 1-cell c_1 as above is Street–invertible if there exist a 1-cell c_1^* and two isomorphic 2-cells $i : 1_{c_0} \to c_1 \circ^0 c_1^*$ and $e : c_1^* \circ^0 c_1 \to 1_{c_0'}$, i.e. such that there exist i^* and e^* with $1_{1_{c_0}} = i \circ^1 i*$, $i^* \circ^1 i = 1_{c_1 \circ^0 c_*}$, $1_{c_1^* \circ^0 c_1} = e \circ^1 e^*$ and $e^* \circ^1 e = 1_{1_{c_0'}}$. This is indeed the usual definition of equivalences in a 2-category. In a 3-category $\mathbb{C}$, a 1-cell c_1 as above is Street–invertible if there exist a 1-cell c_1^* and two 2-equivalences $i : 1_{c_0} \to c_1 \circ^0 c_1^*$ and $e : c_1^* \circ^0 c_1 \to 1_{c_0'}$, i.e. such that there exist i^* and e^* and isomorphic 3-cells $\eta_i : 1_{1_{c_0}} \to i \circ^1 i*$, $\varepsilon_i : i^* \circ^1 i \to 1_{c_1 \circ^0 c_*}$, $\eta_e : 1_{c_1^* \circ^0 c_1} \to e \circ^1 e^*$ and $\varepsilon_e : e^* \circ^1 e \to 1_{1_{c_0'}}$. This gives what is commonly called a 3-equivalence.

The main point of this paper is showing that the two notions of equivalences of Definition 3.2 and of Definition 4.1 coincide. This is summarized by the following

Theorem 4.2. *Let $\mathbb{C}$ be a n-category, and let $c_1 : c_0 \to c_0'$ be an arrow of $\mathbb{C}$. Then c_1 is an equivalence if, and only if, it is Street–invertible.*

An inductive proof will be given in the following pages. One gets immediately the following two

Corollary 4.3. *Let $c_1 : c_0 \to c_0'$ be a weakly invertible 1-cell (that is, according to Definition 3.2). Then for every object c_0'' of $\mathbb{C}$, the morphism*

$$- \circ c_1 : \mathbb{C}_1(c_0'', c_0) \to \mathbb{C}_1(c_0'', c_0')$$

is a weak equivalences of $(n-1)$-categories.

Proof. Since c_1 is weakly invertible, it is also Street–invertible, and for the *"if" part* of the proof of Theorem 4.2, one gets that both $- \circ c_1$ and $c_1 \circ -$ are weak equivalences of $(n-1)$-categories. □

Corollary 4.4. *A strict n-category C is a n-groupoid according to Kapranov and Voevodsky [KV91] if, and only if, it is a n-groupoid according to Street [Str87].*

Proof. The two notions of n-groupoid are based upon the two notions of weak inverses proved to be equivalent by Theorem 4.2. □

In [SKV09] this notion of n-groupoid is used to study exact sequences of n-functors.

4.1 Case analysis: dimension 2

Before digging into the proof of the theorem, let us have a glance at the "if" part, that is the less obvious one, in dimension 2. Notice that only for this section we simplify notation by using juxtaposition for all compositions.
Let $\mathbb{C}$ be a 2-category, and let $c_1 : c_0 \to c_0'$ be Street–invertible. We want to show that 0-compositions with c_1 induces categorical equivalences on the hom-categories obtained by localizing.

For c_1 being Street–invertible amounts to the existence of

$$c_1^* : c_0' \to c_0, \quad i : 1_{c_0} \to c_1 c_1^*, \quad i^* : c_1 c_1^* \to 1_{c_0},$$
$$e : c_1^* c_1 \to 1_{c_0'}, \quad e^* : 1_{c_0'} \to c_1^* c_1;$$

such that

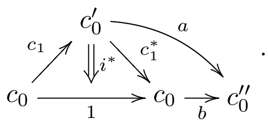
$$i\, i^* = 1, \quad i^*\, i = 1,$$
$$e^*\, e = 1, \quad e\, e^* = 1.$$

For any chosen object c_0'' of $\mathbb{C}$, let us consider the functor

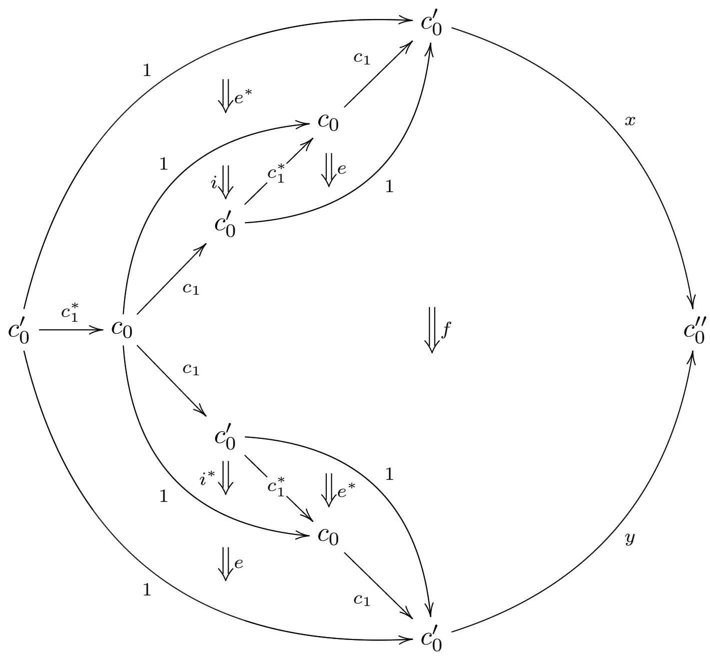
$$c_1 - : \mathbb{C}_1(c_0', c_0'') \to \mathbb{C}_1(c_0, c_0'')$$

(right 0-composition is dealt similarly). We claim it is an equivalence of categories.

Proof. Essentially surjective. Let $c_0 \xrightarrow{b} c_0''$ be given. We exhibit an $c_0' \xrightarrow{a} c_0''$ such that $c_1 a \cong b$. To this end we let $a = c_1^* b$, and the desired isomorphism is

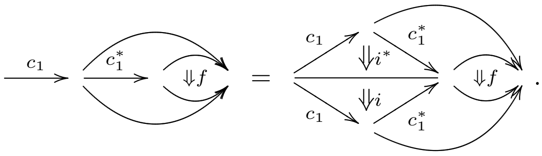
the composition  .

Full. Let the 2-cell $c_0 \underset{c_1 y}{\overset{c_1 x}{\Downarrow f}} c_0''$ be given. We exhibit a 2-cell $c_0' \underset{y}{\overset{x}{\Downarrow g}} c_0''$ such that $c_1 g = f$. To this end we let g be the composition below

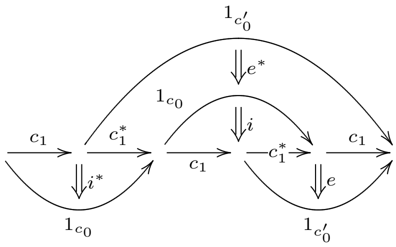


Let us compute $c_1 g$. First of all we observe that as far as have the composite

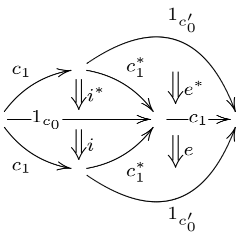
$c_1c_1^*$, I can freely introduce the 2-identity $i\,i^*$, diagrammatically



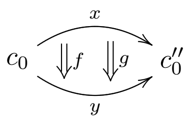
By the symmetry of the diagram, it suffices to study its upper part, in order to show that it reduces to the identity. This will make the composition equal to f. In fact, the upper wing can be redrawn as

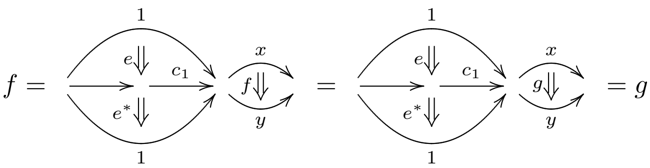


If we slide one over the other the two identities of c_0 in the diagram above, we obtain



Here the 2-cells are mutually inverse isomorphisms, hence they cancel and thus yielding the desired result.

Faithful. Finally, let two parallel 2-cells  be given. We want to show that if $c_1 f = c_1 g$ then $f = g$. For this, simply follow the chain of equality below:



where the hypothesis justifies the second equality. □

5 The proof of Theorem 4.2

This section will take us through the proof of the main theorem. This is quite a long proof, and not really straightforward, at least at the first sight. Nevertheless it is worth to read it fully, as the techniques developed therein will reveal much of the rich structure hiding behind the definitions.

5.1 The "only–if" part

First let us notice that this implication is certainly more expectable than the other. In fact, carefully following a classical categorical argument (the Yoneda embedding), it is possible to show that if a 1-cell is weakly invertible, then it has left and right inverses, i.e. it is Street–invertible.
By induction. For $n = 0$ there is nothing to prove, while for $n = 1$ the proof is an easy consequence of Yoneda lemma. Hence let us fix an integer $n > 1$, and consider consider a 1-cell $c_1 : c_0 \to c'_0$ of $\mathbb{C}$, weakly invertible according to Definition 3.2. Specializing the definition, we get that the 0-composition

$$c_1 \circ - : \mathbb{C}(c'_0, c_0) \to \mathbb{C}(c_0, c_0)$$

is indeed a weak equivalence of $(n-1)$-categories. This produces an essential pre-image of the "object" 1_{c_0}, namely a pair $(c_1^*, \eta : 1_{c_0} \Rightarrow c_1 c_1^*)$, with η weakly invertible. Induction hypothesis immediately implies that η is also Street–invertible, this meaning c_1^* is indeed a right Street–inverse of c_1.
Now we claim that the same c_1^* is also a left Street–inverse. In fact, let us consider the Street invertible 2-cell $\omega = \eta \circ^0 c_1 : c_1 \Rightarrow c_1 c_1^* c_1$. Of course its domain can be seen as the composition $c_1 1_{c'_0}$. Consequently it is possible to get an essential pre-image of ω by means of the weak equivalence of $(n-2)$-categories:

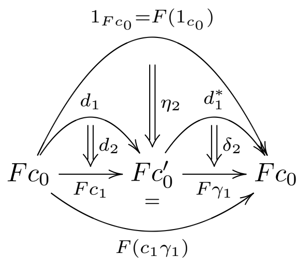
$$[c_1 \circ -]_1^{1_{c'_0}, c_1^* c_1} : [\mathbb{C}(c_0, c'_0)]_1(1_{c'_0}, c_1^* c_1) \to [\mathbb{C}(c_0, c'_0)]_1(c_1, c_1 c_1^* c_1).$$

In this way we produce a pair $\varepsilon : 1_{c'_0} \Rightarrow c_1^* c_1, \Lambda : \omega \Rrightarrow c_1 \circ^0 \varepsilon$. In particular, ε is Street invertible, as a consequence of the following lemma, thus showing that c_1^* is also a left Street–inverse of c_1. This proves our claim, and concludes the proof.

Lemma 5.1. *Let $F : \mathbb{C} \to \mathbb{D}$ be a weak equivalence of n-categories, and let $d_1 : Fc_0 \to Fc'_0$ be a Street–invertible 1-cell of $\mathbb{D}$, then every essential counter-image of d_1 is itself Street-invertible in $\mathbb{C}$.*

Proof. We will prove the statement by induction. For $n = 0$ the lemma holds trivially. More interestingly for $n = 1$, a fully faithful functor reflect isomorphisms, i.e. Street–invertible arrows. Hence suppose we are given an integer $n > 1$.
For every $d_1 : Fc_0 \to Fc'_0$ condition (1) in Definition 3.1 ensures there exists a pair $c_1 : c_0 \to c'_0$, $d_2 : d_1 \Rightarrow Fc_1$, with d_2 weakly invertible. We claim that c_1 is Street–invertible.
In order to prove the claim we have to exhibit a Street–inverse $\gamma_1 : c'_0 \to c_0$ and the two witnesses (Street–invertible 2-cells) $\gamma_2 : 1_{c_0} \Rightarrow c_1\gamma_1$ and $\gamma'_2 : \gamma_1 c_1 \Rightarrow 1_{c'_0}$.
To start with, let us consider a Street-inverse of d_1, let us call it $d_1^* : Fc'_0 \to Fc_0$, together with their witnesses $\eta_2 : 1_{Fc_0} \Rightarrow d_1 d_1^*$ and $e_2 : d_1^* d_1 \Rightarrow 1_{Fc'_0}$. Notice that witnesses are themselves Street-invertible, thus there are e_2^* and η_2^* etc. Let us consider an essential pre-image of d_1^*, i.e. a pair $\gamma_1 : c'_0 \to c_0$, $\delta_2 : d_1^* \Rightarrow F\gamma_1$,

with δ_2 weakly invertible. This allows us to fill the diagram below:

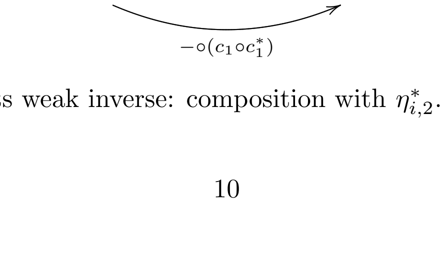


Now d_2 and δ_2 are Street–invertible, so that also $m_2 := \eta_2 \circ^1 (d_2 \circ_0 \delta_2)$ is. Now,

$$F_2^{1_{c_0}, c_1\gamma 1} : \mathbb{C}(c_0, c_0)(1_{c_0}, c_1\gamma_1) \to \mathbb{D}(Fc_0, Fc_0)(F1_{c_0}, Fc_1\gamma_1)$$

is a localization of a weak equivalence, that is a weak equivalence itself, so that one can find a pair $n_2 : 1_{c_0} \Rightarrow c_1\gamma_1$, $m_2 \Rrightarrow Fn_2$. Finally, by induction hypothesis n_2 is Street–invertible. In fact, it is the essential pre-image via a 2-localization of F (= a 1-localization of F_1) of a Street–invertible 2-cell, namely m_2. In order to conclude the proof we shall observe that the same argument applies when considering the composition $\gamma_1 c_1$. □

Remark 5.2. Now that it is clear that weakly invertible cells have indeed inverses, one can ask the same question for morphisms. This is critical in order to justify or modify our terminology. Indeed weakly invertible cells give equivalence relations, for instance, weakly invertible 1-cells establish an equivalence relation in the class of the objects of a n-category. Differently weak equivalences of n-categories do not give any equivalence relation in the class of n-categories. In fact it is easy to show that the identity morphism of an n-category is indeed a weak equivalence, and that the composition of two weak equivalences is again a weak equivalence, thus giving a reflexive and transitive relation. But this relation is in general not symmetric, as the existence of weak equivalence n-functor between two n-categories does not imply the existence of an inverse equivalence n-functor. This would be the case (under the assumption of the axiom of choice) if we consider a weaker notion of morphism of n-categories, but this would take us far from the aims of the present work, hence we leave it for future developing.
Nevertheless, an important class of weak equivalences is indeed invertible in our setting, namely those that are given by left and right compositions with a weakly invertible cell. In fact for a weakly invertible 1-cell $c_1 : c_0 \to c_0'$, the morphisms of $(n-1)$-categories $- \circ^0 c_1$ has an inverse given by $- \circ^0 c_1^*$. Explicitly, by associativity their composition amounts to the n-functor $- \circ^0 (c_1 \circ^0 c_1^*)$. Now composition with $\eta_{i,2}$ gives the desired 2-morphism: for any object c_0'' in $\mathbb{C}$ the natural n-transformation



Similarly one gets its weak inverse: composition with $\eta_{i,2}^*$.

5.2 Notation for inverses

In order to deal with inductive inverses used along the proof in the next section, we switch to a more practical scalable notation for Street–inverses and units we will need. At the base of the inductive process there is a 1-cell c_1, and one chosen inverse c_1^*:

$$c_1 : c_0 \longrightarrow c_0' , \qquad c_1^* : c_0' \longrightarrow c_0 .$$

Then we define the first witnesses of c_* being inverse of c_1:

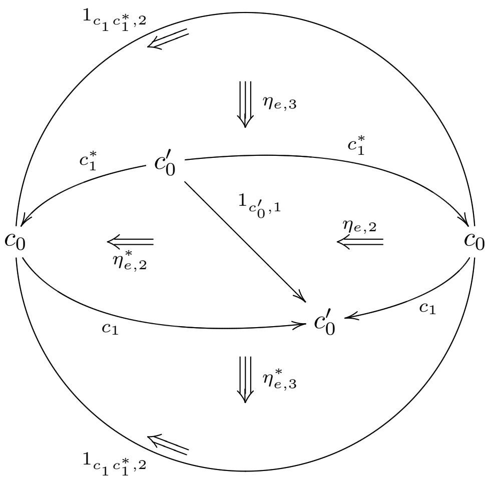
$$\eta_{e,2} : c_1^* \circ^0 c_1 \longrightarrow 1_{c_0',1} , \qquad \eta_{i,2} : 1_{c_0,1} \longrightarrow c_1 \circ^0 c_1^* .$$

where $1_{c_0,1} = id_{c_0} : c_0 \to c_0$ is the identity 1-cell over the object c_0. Then, for every k we define the k^{th} witnesses of c_1 being Street–invertible

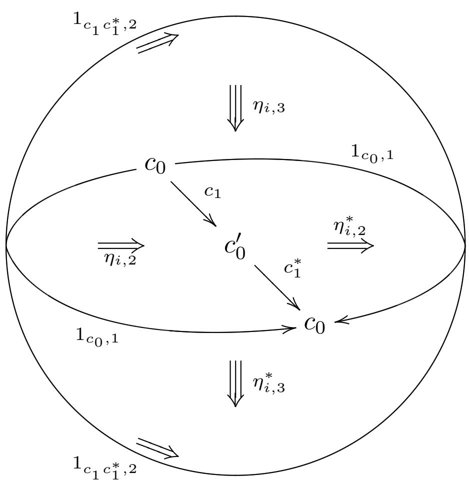
$$\eta_{e,k+1} : 1_{c_1^* c_1,k} \longrightarrow \eta_{e,k} \circ^{k-1} \eta_{e,k}^* , \qquad \eta_{i,k+1} : 1_{c_0,k} \longrightarrow \eta_{i,k} \circ^{k-1} \eta_{i,k}^* .$$

where $1_{x_h,k}$ is the h-cell x, seen as a k-identity over x, with $h < k$.

The situation in dimension 3 may be easily visualized with the help of globes, in order to lead the intuition in higher dimensions. In the pictures below objects are points, arrows are edges, two-cells are surfaces and three-cells are volumes, orientation being described by directional arrows on those. The two globes describe the left and the right inverses of a 1-cell $c_1 : c_0 \to c_0'$.



(2)



(3)

5.3 The "if" part

The implication vacuously holds for $n = 0$. For $n = 1$, a weakly invertible 1-cell is just an isomorphic one, then it is clear that composition with it induces isomorphisms between the hom–sets. Hence we can well suppose $n > 1$. Let a Street–invertible 1-cell $c_1 : c_0 \to c_0$ be given. We want to prove that for any other object c''_0 of $\mathbb{C}$, the morphisms of $(n-1)$-categories

$$c_1 \circ^0 - : \mathbb{C}_1(c'_0, c''_0) \to \mathbb{C}_1(c_0, c''_0) \qquad - \circ^0 c_1 : \mathbb{C}_1(c''_0, c_0) \to \mathbb{C}_1(c''_0, c'_0)$$

are weak equivalences, i.e. we are going to show that:

- ⋆ $c_1 \circ^0 - : \mathbb{C}_1(c'_0, c''_0) \to \mathbb{C}_1(c_0, c''_0)$ is h-surjective,
- ⋆ for any pair of 1-cells $\gamma_1, \gamma'_1 : c'_0 \to c''_0$, the localization

  $$[c_1 \circ^0 -]_1^{\gamma_1,\gamma'_1} : \mathbb{C}_2(\gamma_1, \gamma'_1) \to \mathbb{C}_2(c_1 \circ^0 \gamma_1, c_1 \circ^0 \gamma'_1)$$

  is h-surjective,
- ⋆ for any pair of 2-cells $\gamma_2, \gamma'_2 : \gamma_1 \to \gamma'_1$, the localization

  $$[c_1 \circ^0 -]_2^{\gamma_2,\gamma'_2} : \mathbb{C}_3(\gamma_2, \gamma'_2) \to \mathbb{C}_3(c_1 \circ^0 \gamma_2, c_1 \circ^0 \gamma'_2)$$

  is h-surjective,

and so on, up to $n-1$ where this means *full*, and for n where this means *faithful*. At each step h-surjectivity will be proved indirectly. The idea is that a morphism $P : \mathbb{A} \to \mathbb{B}$ is h-surjective if one can find a morphism $Q : \mathbb{B} \to \mathbb{A}$ and an equivalence 2-morphism $\alpha : QP \Rightarrow Id_{\mathbb{B}}$. Due to the symmetry of the notion of Street-invertible cells, the proof that the morphisms $- \circ c_1$ are weak equivalences of $(n-1)$-categories is completely analogous, and hence it is left to the reader.
Let us notice that an increasing number of steps are necessary at each dimension: two steps of h-surjectivity are necessary in dimension 1, namely the usual surjectivity and the injectivity, three in dimension two, etc. and all these are

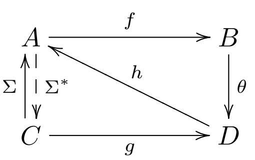
somehow related in a kind of inductive network, since localizing is a process that lowers the dimension. The picture comes clearer if we organize them in a triangular matrix:

1	2	3	4	$\cdots$	k	$\cdots$	n
$\mathfrak{D}^1_1$	$\mathfrak{D}^2_1$	$\mathfrak{D}^3_1$	$\mathfrak{D}^4_1$	$\cdots$	$\mathfrak{D}^k_1$	$\cdots$	$\mathfrak{D}^n_1$
$\mathfrak{D}^2_1$	$\mathfrak{D}^2_2$	$\mathfrak{D}^3_2$	$\mathfrak{D}^4_2$	$\cdots$	$\mathfrak{D}^k_2$	$\cdots$	$\mathfrak{D}^n_2$
	$\mathfrak{D}^2_3$	$\mathfrak{D}^3_3$	$\mathfrak{D}^4_3$	$\cdots$	$\mathfrak{D}^k_3$	$\cdots$	$\mathfrak{D}^n_3$
		$\mathfrak{D}^3_4$	$\mathfrak{D}^4_4$	$\cdots$	$\mathfrak{D}^k_4$	$\cdots$	$\mathfrak{D}^n_4$
			$\mathfrak{D}^4_5$	$\cdots$	$\mathfrak{D}^k_5$	$\cdots$	$\mathfrak{D}^n_5$
				$\ddots$	$\vdots$		$\vdots$
					$\mathfrak{D}^k_{k+1}$	$\cdots$	$\mathfrak{D}^n_{k+1}$
						$\ddots$	$\vdots$
							$\mathfrak{D}^n_{n+1}$

where the symbol $\mathfrak{D}^k_\ell$ represents the h-surjectivity (up to $(k-\ell)$-cells) of the ℓ–localization of the morphism $c_1 \circ^0 -$, with c_1 an arrow of the k-category $\mathbb{C}$. Induction will run over the variables k and ℓ. More precisely the induction hypothesis in order to prove $\mathfrak{D}^k_\ell$ is that $\mathfrak{D}^x_y$ holds for $x < k$, and for $x = k$ if $y < \ell$.

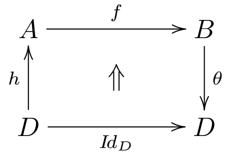
5.4 The abstract scheme

Having reduced the proof to showing different kind of surjectivity, it is reasonable to look for a general scheme in which the different proofs of each of the $\mathfrak{D}^k_\ell$ would fit.
To this purpose, let us suppose we are given the diagram below, where we want to prove that the morphism $g : C \to D$ is h-surjective:

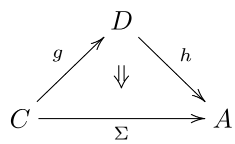


$(*)$

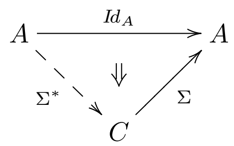
For the reasons stated above, this can be achieved by exhibiting an equivalence 2-morphism $h\Sigma^* g \Rightarrow Id_D$ (thesis). In order to get the last, it is easier to past three simpler 2-morphisms, represented by three diagrams below (hypothesis):



(i)



(ii)



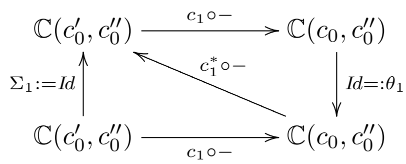
(iii)

Then the proof is the composition:

$$h\Sigma^* g = h\Sigma^* g Id_D \overset{(i)}{\Longrightarrow} h\Sigma^* g h f \theta \overset{(ii)}{\Longrightarrow} h\Sigma^* \Sigma f \theta \overset{(iii)}{\Longrightarrow} h f \theta \overset{(i)^*}{\Longrightarrow} Id_D \ .$$

In the diagram above, (i) is a 2-morphism obtained by 0-composing with a Street-invertible 2-cell; hence $(i)^*$ is the 2-morphism obtained by 0-composing with one of its Street–inverses.
Applying this scheme to our specific situation will result in the construction three diagrams for each pair of positive integers (k, ℓ) with $\ell \leq k+1$. Those will be denoted by $\mathfrak{D}^k_\ell(i), \mathfrak{D}^k_\ell(ii)$ and $\mathfrak{D}^k_\ell(iii)$, while the general scheme diagram will be denoted by $\mathfrak{D}^k_\ell(*)$.
Now, although the induction process described by the triangular matrix run *up* to *down* (localizing) and then *left* to *right* (raising dimension), our description will be by rows, as diagrams with fixed ℓ are identical, even if they are to be interpreted in different dimensions.

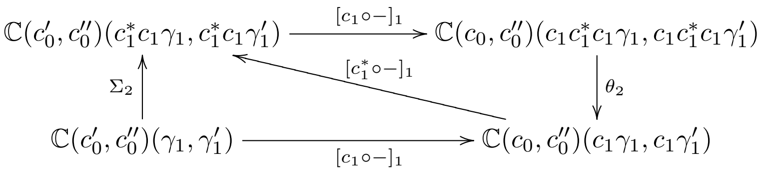
Hence diagram $\mathfrak{D}^k_1(*)$ is given by



(4)

Let us notice that the identity Σ_1 is really the degenerate case $1_{c'_0,1} \circ^0 -$. For $k = 1$ this a diagram of (hom)–sets and maps, hence its commutativity is required, while for higher dimensions, it will commute up to suitable 2-morphisms.

Diagram $\mathfrak{D}^k_2(*)$ is given by localizing the *zig–zag* of $\mathfrak{D}^k_1(*)$, for any pair $\gamma_1, \gamma'_1 : c'_0 \to c''_0$:



(5)

Here Σ_2 is the left/right 1-composition $\eta_{e,2}\gamma_1 \circ^1 - \circ^1 \eta^*_{e,2}\gamma'_1$. With the help of the interchange property of compositions, it can be rewritten as $(\eta_{e,2} \circ^1 \eta^*_{e,2}) \circ^0 -$. Referring to globes diagrams (2) and (3), this can be seen as a 0-composition with a 2-dimensional disk, for $k = 2$ this is again an identity, where for the 2-cell $\eta_{e,2}$ being an equivalence means it is just an isomorphism. On the other side, θ_2 is the left/right 1-composition $\eta_{i,2}c_1\gamma_1 \circ^1 - \circ^1 \eta_{i,2}c_1\gamma'_1$ that cannot be simplified as Σ_2 before.
As it often happens in higher category theory, the investigation of the 3–dimensional case is crucial in order to start induction properly: if something nasty is going to happen in higher dimensions, then it is likely to start in dimension three!
Localizing the *zig–zag* of $\mathfrak{D}^k_2(*)$, for any pair of 2-cells $\gamma_2, \gamma'_2 : \gamma_1 \to \gamma'_1$ gives the

backbone of diagram $\mathfrak{D}_3^k(*)$:

$$\begin{array}{ccc} \mathbb{C}_3(c_1^*c_1\gamma_2, c_1^*c_1\gamma_2') & \xrightarrow{[c_1\circ -]_2} & \mathbb{C}_3(c_1c_1^*c_1\gamma_2, c_1c_1^*c_1\gamma_2') \\ \Sigma_3 \uparrow & \nwarrow^{[c_1^*\circ -]_2} & \downarrow \theta_3 \\ \mathbb{C}_3(\gamma_2, \gamma_2') & \xrightarrow[{[c_1\circ -]_2}]{} & \mathbb{C}_3(c_1\gamma_2, c_1\gamma_2')\,. \end{array} \tag{6}$$

Notice that we use a simplified notation for hom's, e.g. $\mathbb{C}_3(c_1^*c_1\gamma_2, c_1^*c_1\gamma_2')$ for $\mathbb{C}(c_0', c_0'')(c_1^*c_1\gamma_1, c_1^*c_1\gamma_1')(c_1^*c_1\gamma_2, c_1^*c_1\gamma_2')$.
On the contrary, the connecting morphisms Σ_3 and θ_3 deserve some analysis. In fact we could not simply localize Σ_2 as it would take us in a different codomain, from the one reached by localizing $[c_1 \circ -]_1$ composed $[c_1^* \circ -]_1$. In order to fill the gap, it is necessary to connect those by the dashed morphism in the diagram below

$$\begin{array}{c} \mathbb{C}_3(c_1^*c_1\gamma_2, c_1^*c_1\gamma_2') \\ \uparrow \scriptstyle \eta_{e,3}\gamma_2\circ^2 - \circ^2\eta_{e,3}^*\gamma_2' \\ \mathbb{C}_3((\eta_{e,2} \circ^1 \eta_{e,2}^*)\gamma_2, (\eta_{e,2} \circ^1 \eta_{e,2}^*)\gamma_2') \\ \uparrow \scriptstyle [\Sigma_2]_1 \\ \mathbb{C}_3(\gamma_2, \gamma_2') \end{array}$$

This composition defines Σ_2, that by interchange laws it can be rewritten as

$$(\eta_{e,3} \circ^2 \eta_{e,3}^*) \circ^0 - \quad .$$

For the morphism θ_3 the problem is similar, similar the solution: the required connecting morphism is the dashed arrow below:

$$\begin{array}{c} \mathbb{C}_3(c_1c_1^*c_1\gamma_2, c_1c_1^*c_1\gamma_2') \\ \downarrow \scriptstyle [\theta_2]_1 \\ \mathbb{C}_3(\eta_{i,2}c_1\gamma_1 \circ^1 c_1c_1^*c_1\gamma_2 \circ^1 \eta_{i,2}^*c_2\gamma_1', \eta_{i,2}c_1\gamma_1 \circ^1 c_1c_1^*c_1\gamma_2' \circ^1 \eta_{i,2}^*c_1\gamma_1') \\ \| \scriptstyle (\spadesuit) \\ \mathbb{C}_3((\eta_{i,2} \circ^1 \eta_{i,2}^*)c_1\gamma_2, (\eta_{i,2} \circ^1 \eta_{i,2}^*)c_1\gamma_2') \\ \downarrow \scriptstyle \eta_{i,3}c_1\gamma_2\circ^2 - \circ^2\eta_{i,3}^*c_1\gamma_2' \\ \mathbb{C}_3(c_1\gamma_2, c_1\gamma_2') \end{array}$$

where the equality $(\spadesuit)$ is guaranteed by the Interchange Lemma 5.3. Let us observe that in this case it is not possible to rewrite it as a 0-composition with a globe, as for Σ_3.
Finally, for a generic $\ell = \lambda$ one has the diagram $\mathfrak{D}_\lambda^k(*)$:

$$\begin{array}{ccc} \mathbb{C}_\lambda(c_1^*c_1\gamma_{\lambda-1}, c_1^*c_1\gamma_{\lambda-1}') & \xrightarrow{[c_1\circ -]_{\lambda-1}} & \mathbb{C}_\lambda(c_1c_1^*c_1\gamma_{\lambda-1}, c_1c_1^*c_1\gamma_{\lambda-1}')\,. \\ \Sigma_\lambda \uparrow & \nwarrow^{[c_1^*\circ -]_{\lambda-1}} & \downarrow \theta_\lambda \\ \mathbb{C}_\lambda(\gamma_{\lambda-1}, \gamma_{\lambda-1}') & \xrightarrow[{[c_1\circ -]_{\lambda-1}}]{} & \mathbb{C}_\lambda(c_1\gamma_{\lambda-1}, c_1\gamma_{\lambda-1}') \end{array} \tag{7}$$

where vertical morphisms are defined inductively:

$$\mathbb{C}_\lambda(\gamma_{\lambda-1},\gamma'_{\lambda-1}) \xrightarrow[{[\Sigma_{\lambda-1}]_1}]{} [\star]_\lambda \xrightarrow[\eta_{e,\lambda}\gamma_{\lambda-1}\circ^{\lambda-1} - \circ^{\lambda-1}\eta^*_{e,\lambda}\gamma'_{\lambda-1}]{} \mathbb{C}_\lambda(c_1^*c_1\gamma_{\lambda-1}, c_1^*c_1\gamma'_{\lambda-1})$$

$$\mathbb{C}_\lambda(c_1c_1^*c_1\gamma_{\lambda-1}, c_1c_1^*c_1\gamma'_{\lambda-1}) \xrightarrow[{[\theta_{\lambda-1}]_1}]{} [\bullet]_\lambda \xrightarrow[\eta_{i,\lambda}c_1\gamma_{\lambda-1}\circ^{\lambda-1} - \circ^{\lambda-1}\eta^*_{i,\lambda}\gamma'_{\lambda-1}]{} \mathbb{C}_\lambda(c_1\gamma_{\lambda-1}, c_1\gamma'_{\lambda-1})$$

In fact these two pair of morphisms are composable. We examine the $[\bullet]_\lambda$ in detail. By induction, the codomain of the first morphism is the $(n-\lambda)$-category

$$\mathbb{C}_\lambda\Big(\eta_{i,\lambda-1}c_1\gamma_{\lambda-2}\circ^{\lambda-2}(\eta_{i,\lambda-2}\circ^{\lambda-3}\eta^*_{i,\lambda-2})c_1\gamma_{\lambda-1}\circ^{\lambda-2}\eta^*_{i,\lambda-1}c_1\gamma'_{\lambda-2}, \qquad (8)$$

$$\eta_{i,\lambda-1}c_1\gamma_{\lambda-2}\circ^{\lambda-2}(\eta_{i,\lambda-2}\circ^{\lambda-3}\eta^*_{i,\lambda-2})c_1\gamma'_{\lambda-1}\circ^{\lambda-2}\eta^*_{i,\lambda-1}c_1\gamma'_{\lambda-2}\Big)$$

while by definition the domain on the second is

$$\mathbb{C}_\lambda\big((\eta_{i,\lambda-1}\circ^{\lambda-2}\eta^*_{i,\lambda-1})c_1\gamma_{\lambda-1}, (\eta_{i,\lambda-1}\circ^{\lambda-2}\eta^*_{i,\lambda-1})c_1\gamma'_{\lambda-1}\big) \qquad (9)$$

Their equality is just a straightforward application of the first statement of Lemma 5.3.
Concerning $[\star]_\lambda$, a similar discussion can be carried on, by the second statement of the Lemma.

Lemma 5.3. *Let cells c's λ's and η's be given as above. Then the following two equations hold:*

$$\eta_{i,\lambda-1}c_1\gamma_{\lambda-2}\circ^{\lambda-2}(\eta_{i,\lambda-2}\circ^{\lambda-3}\eta^*_{i,\lambda-2})c_1\gamma_{\lambda-1}\circ^{\lambda-2}\eta^*_{i,\lambda-1}c_1\gamma'_{\lambda-2} =$$

$$= (\eta_{i,\lambda-1}\circ^{\lambda-2}\eta^*_{i,\lambda-1})c_1\gamma_{\lambda-1} \qquad (10)$$

$$\eta_{e,\lambda-1}\gamma_{\lambda-2}\circ^{\lambda-2}(\eta_{e,\lambda-2}\circ^{\lambda-3}\eta^*_{e,\lambda-2})\gamma_{\lambda-1}\circ^{\lambda-2}\eta^*_{e,\lambda-1}\gamma'_{\lambda-2} =$$

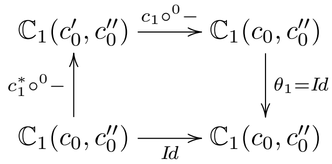
$$= (\eta_{e,\lambda-1}\circ^{\lambda-2}\eta^*_{e,\lambda-1})\gamma_{\lambda-1} \qquad (11)$$

Proof. It is sufficient to apply functoriality of compositions, a.k.a. interchange rules of n-categories. □

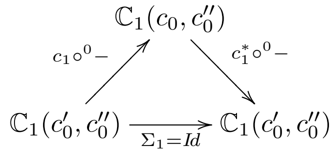
5.5 The base of the induction

The base of the induction can be deduced directly, but it is useful to show at least the three self–explaining diagrams in dimension 1. Let us observe that since an arrow of a category $\mathbb{C}$ is an equivalence when it is an isomorphism, the composites like $c_1^*c_1$ are identities. Moreover diagrams commutes strictly, as the only natural transformation between (hom)–set–theoretical maps are the identical ones.

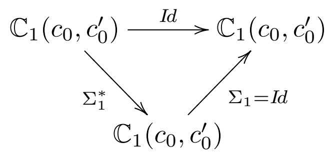
$\mathfrak{D}_1^1(i)$:



$\mathfrak{D}_1^1(ii)$:



$\mathfrak{D}_1^1(iii)$:

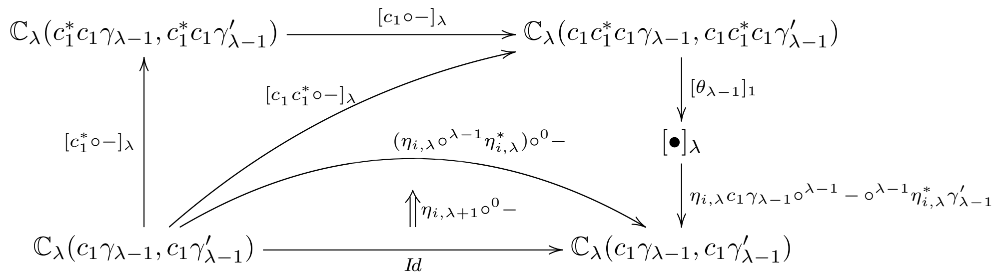


5.6 Proof of $\mathfrak{D}_\ell^k(i)$

From now on, suppose we are given cells:

$$\gamma_{\lambda-1}, \gamma'_{\lambda-1} : \gamma_{\lambda-2} \to \gamma'_{\lambda-2} : \cdots : \gamma_1 \to \gamma'_1 : c_0' \to c_0''$$

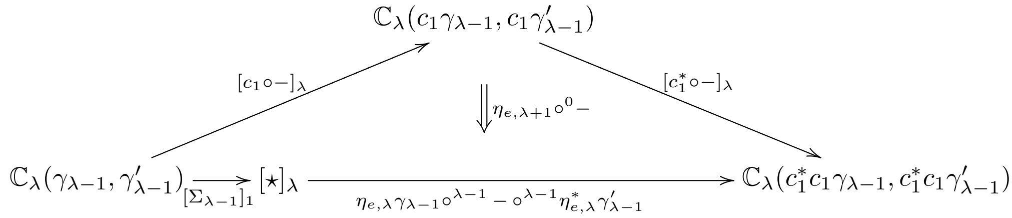
Let us consider the diagram below:



By induction, one can suppose that the composition $[c_1 c_1^* \circ -]_\lambda \cdot [\theta_{\lambda-1}]_1$ equals to the morphism $(\eta_{i,\lambda-1} \circ^{\lambda-2} \eta^*_{i,\lambda-1}) \circ^0 -$. Then $(\lambda-1)$-composing with $\eta_{i,\lambda} c_1 \gamma_{\lambda-1}$ on the left and with $\eta^*_{i,\lambda} \gamma'_{\lambda-1}$ on the right can be seen as pasting two hemispheres on a disk of the same diameter (always refer to diagrams (2) and (3) for an intuition), hence by interchange the whole turns into a 0-composition with the globe $(\eta_{i,\lambda} \circ^{\lambda-1} \eta^*_{i,\lambda})$. This explains the commutativity of the empty part of the diagram, that so can be filled with the 2-morphism $\eta_{i,\lambda+1} \circ^0 -$.

5.7 Proof of $\mathfrak{D}^k_\ell(ii)$

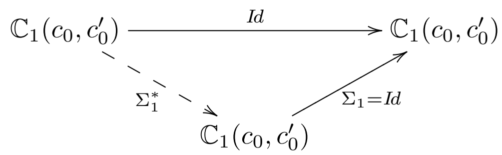
Concerning the diagrams of type (ii), the proof is similar but simpler. Let us consider the diagram



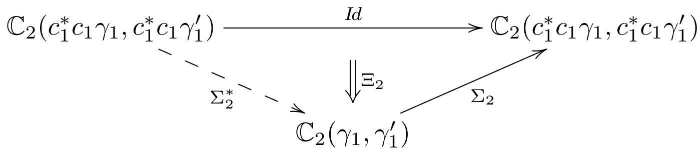
By induction one can deduce that the composition at the base of the triangle is the indeed the morphism $(\eta_{e,\lambda} \circ^{\lambda-1} \eta e, \lambda^*) \circ^0 -$, i.e. a 0-composition with a globe. The 2-morphism follows as before.

5.8 Proof of $\mathfrak{D}^k_\ell(iii)$

The construction of type-(iii) diagrams is slightly more delicate. In fact here not only equations come from induction, but even the *existence* of some morphisms, hence it will be detailed in low dimensions.
The triangle $\mathbb{D}^k_1(iii)$



is commutative. In this case, the inverse morphism Σ_1^* is just the (strict) inverse of an identity, namely the identity itself.
Next, diagram $\mathbb{D}^k_2(iii)$ is given below:



The morphism Σ_2 is a left/right 1-composition with $\eta_{e,2}\gamma_1$, resp. $\eta^*_{e,2}\gamma'_1$. Then it has a weak–inverse given by exchanging left with right composite. Moreover, since

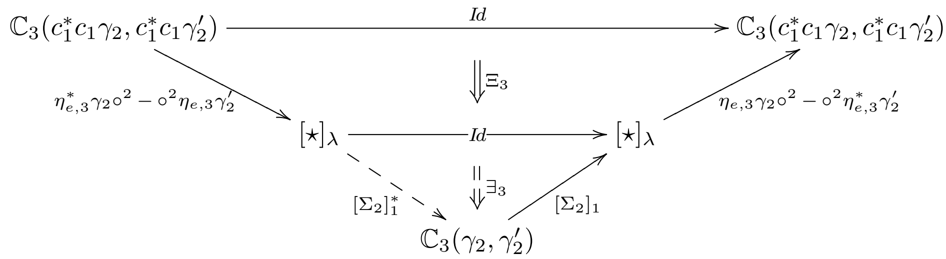
$$\begin{aligned} \Sigma_2 &= \eta_{e,2}\gamma_1 \circ^1 - \circ^1 \eta^*_{e,2}\gamma'_1 \\ \Sigma_2^* &= \eta^*_{e,2}\gamma_1 \circ^1 - \circ^1 \eta_{e,2}\gamma'_1 \end{aligned}$$

one can compute their composition:

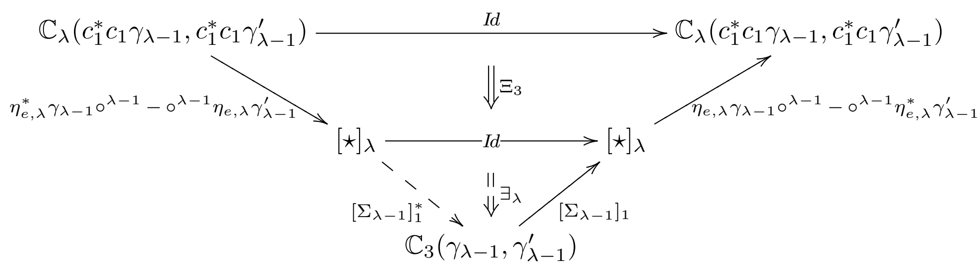
$$\Sigma_2^* \cdot \Sigma_2 = (\eta_{e,2} \circ^1 \eta^*_{e,2})\gamma_1 \circ^1 - \circ^1 (\eta_{e,2} \circ^1 \eta^*_{e,2})\gamma'_1$$

and we can define a suitable 2-morphism $\Xi_2 = \eta_{e,3}\gamma_1 \circ^2 - \circ^2 \eta_{e,3}\gamma'_1$ that complete the picture.

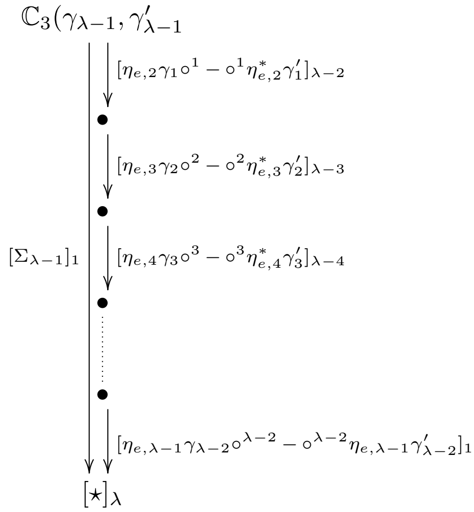
The 3-dimensional case is represented in diagram $\mathbb{D}_3^k(iii)$ below, and it gives the model for the general situation:



Our investigations in dimension 2 suggests that the trapezoidal area can be filled with the 2-morphism $\Xi_3 = \eta_{e,4}\gamma_2 \circ^3 - \circ^3 \eta_{e,4}\gamma_2'$, the triangle below must be constructed first. In fact what we are doing is finding a left weak–inverse to the morphism $[\Sigma_2]_1$. Although this is a localization of the equivalence Σ_2, it cannot just be the localization of its inverse, because in general this would take far from the desired codomain $\mathbb{C}_3(\gamma_2, \gamma_2')$. Thus we must use induction. $[\Sigma_2]_1$ is the localization of a morphism given by two 1-compositions. By the dimensional shift, in the hom–2-categories the 1-composition is precisely their 0-composition. Now the (whole) theorem states that those morphism that are given by a 0-composition with an equivalence–1-cell are weakly invertible. Then there exist $[\Sigma_2]_1^*$ and $\exists_3$ as in the diagram above, and this concludes the 3-dimensional case. Finally we deal with diagram $\mathbb{D}_3^k(iii)$



Again, the 2-morphism Ξ_λ can be defined directly in terms of compositions, i.e. $\eta_{e,\lambda+1}\gamma_{\lambda-1} \circ^\lambda - \circ^\lambda \eta_{e,\lambda+1}\gamma_{\lambda-1}'$. Concerning $[\Sigma_{\lambda-1}]_1$, it has a weak–inverse by induction. In fact, it is the composite of decreasing localizations of compositions of increasing dimension:



From up to down, the first composite has a weak–inverse by $\mathfrak{D}^{k-1}_{\lambda-1}$, the second by $\mathfrak{D}^{k-1}_{\lambda-2}$, and so on up to the $(\lambda-2)^{\text{th}}$, for which the property holds by $\mathfrak{D}^{k-1}_2$.

References

[KV91] M. M. Kapranov and V. A. Voevodsky. ∞-groupoids and homotopy types. *Cahiers Topologie Géom. Différentielle Catég.*, 32(1):29–46, 1991. International Category Theory Meeting (Bangor, 1989 and Cambridge, 1990).

[LMW07] Yves Lafont, Francois Metayer, and Krzysztof Worytkiewicz. A folk model structure on omega-cat, 2007.

[Met08] Giuseppe Metere. *The Ziqqurath of exact sequences of n-groupoids.* PhD thesis, University of Milan, February 2008. (arXiv:0802.0800).

[Sim98] Carlos Simpson. Homotopy types of strict 3-groupoids, 1998.

[SKV09] G. Metere S. Kasangian and E.M. Vital. The ziqqurath of exact sequences of n-groupoids. *submitted*, 2009.

[Str87] Ross Street. The algebra of oriented simplexes. *J. Pure Appl. Algebra*, 49(3):283–335, 1987.

[Str96] R. Street. Categorical structures. *Handbook of Algebra*, 1:529–577, 1996.