Some thoughts on Yoneda's "regular spans" and related notions

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5th Workshop on Categorical Methods in Non-Abelian Algebra, UCL, Louvain-la-Neuve, July 2017.

Overview

- Introduction: Yoneda's regular spans
- The fibred viewpoint
- A non-abelian variation

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Yoneda's spans

In 1960, Nobuo Yoneda published the paper:

N.Yoneda, **On Ext and exact sequences**, *J.Fac.Sci.Univ. Tokyo Sect.I 8*

which provides a description of (equivalences classes of) n-fold extensions

$$0 \to B \to E_n \to \cdots \to E_1 \to A \to 0$$

in a (sufficiently good) additive category $\mathcal{A}$.

Yoneda himself explains his idea in the introduction.

Yoneda's On Ext and exact sequences

"The n-fold extensions over A with kernel B in an additive category $\mathcal{A}$ will be considered as some quantity lying between A and B, or lying over the pair (A, B), which we want to classify to get $\mathbf{Ext}^n(A, B)$. Then, the totally of n-fold extensions in $\mathcal{A}$ is considered as a sort of web spanning pairs of objects. [...] the web gives a certain correspondence between $\mathcal{A}$ and a copy of $\mathcal{A}$ [...] which renders the functorial structure of $\mathbf{Ext}^n(A, B)$."

"In generalizing this situation, we consider a pair of categories $(\mathcal{A}, \mathcal{B})$ and a third category $\mathcal{X}$ together with two covariant functors $S_- : \mathcal{X} \to A$ and $S_+ : \mathcal{X} \to A$, or to the same effect, a covariant functor $S : \mathcal{X} \to \mathcal{A} \times \mathcal{B}$, which we call span over $(\mathcal{A}, \mathcal{B})$."

"Our domain of theory will be abstract categories, and no applications are intended in this paper. They will be found elsewhere."

Yoneda's regular spans

Given a span $S\colon \mathcal{X} \to \mathcal{A} \times \mathcal{B}$, let $S_{(a,b)}$ be the fibre of S over the object (a, b), and let $\bar{S}_{(a,b)} = \pi_0(S_{(a,b)})$.

Question

What axioms on S in order to get $\bar{S}_{(-,-)}$ functorial?

Definition (Yoneda, 1960)

S is a regular span if

- $\mathcal{X} \xrightarrow{S} \mathcal{A} \times \mathcal{B} \xrightarrow{P_0} \mathcal{A}$ is a fibration with enough P_1S-vertical P_0S-cartesian lifts,
- $\mathcal{X} \xrightarrow{S} \mathcal{A} \times \mathcal{B} \xrightarrow{P_1} \mathcal{A}$ is an opfibration with enough P_0S-vertical P_1S-opcartesian lifts.

Theorem (Yoneda, 1960)

*$\bar{S}_{(a,b)}$ defines a functor $\mathcal{A}^{op} \times \mathcal{B} \to$ **Set**.*

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What is a regular span?

Question

What is a regular span S?
Is it just an ad-hoc definition in order to make $\bar{S}_{(a,b)}$ functorial?

Proposition (CMM, 2017)

For a given span $S\colon \mathcal{X} \to \mathcal{A} \times \mathcal{B}$, TFAE:

- *S is a regular span*
- *$S\colon P_0S \to P_0$ is a fiberwise opfibration in* $\mathbf{Fib}(\mathcal{A})$

  i.e. a cartesian functor over $\mathcal{A}$ such that, for any object a of $\mathcal{A}$, its restrictions to the fibres over a are opfibrations:

$$S_a\colon \mathcal{X}_a \to \{a\} \times \mathcal{B} \simeq \mathcal{B}$$

- *$S\colon P_1S \to P_1$ is a fiberwise fibration in* $\mathbf{opFib}(\mathcal{B})$

What is a regular span?

Still, definitions seem to be ad-hoc. . .

How much of such definitions is internal (in a 2-categorical sense)?

Let us consider the following conditions on a commut. diagram in **Cat**:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{P} & \mathcal{M} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle P_0} & \\ & \mathcal{A} & \end{array}$$

(1) P is a morphism in $\mathbf{Fib}(\mathcal{A})$

(2) for every object a of $\mathcal{A}$, the restriction to the fibres

$$\mathcal{X}_a \xrightarrow{P_a} \mathcal{M}_a$$

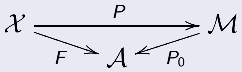
is an opfibration

(3) for every two arrows $\alpha\colon a_1 \to a_2$ and $\mu\colon m_1 \to m_2$ such that $P_0(\mu) = 1_{a_2}$, and any object x of $\mathcal{X}$ such that $P(x) = m_1$, then the canonical comparison $\alpha^*\mu_*x \to \mu_*\alpha^*x$ is an isomorphism.

What is a regular span?

Proposition (CMM, 2017)

For a commutative diagram in **Cat**



- (1)+(2) $\overset{def}{\Leftrightarrow}$ *P fiberwise opfibration* $\Leftrightarrow$ (1) + *P opfibration in* **Cat**$/\mathcal{A}$
- (1)+(2)+(3) $\Leftrightarrow$ *P opfibration in* **Fib**$(\mathcal{A})$
- (2 discr.) $\Leftrightarrow$ *P discrete opfibration in* **Cat**$/\mathcal{A}$
- (1)+(2 discr.) $\Leftrightarrow$ (1)+(2 discr.)+(3) $\Leftrightarrow$ (1)+(2 discr.)+(3 id.) $\Leftrightarrow$ *P discrete opfibration in* **Fib**$(\mathcal{A})$

The proof uses Chevalley criterion [Street 1974, after Gray 1966], for the characterization of (Grothendieck) cloven opfibrations in terms of (normal) pseudo-algebras for a lax idempotent 2-monad.

What is a regular span?

Corollary

For a commutative diagram in **Cat**

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\;\;S\;\;} & \mathcal{A}\times\mathcal{B} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle P_0} & \\ & \mathcal{A} & \end{array}$$

- (1)+(2) *i.e. S fiberwise opfibration* $\Leftrightarrow$ *S regular span*
- (1)+(2)+(3) $\Leftrightarrow$ *S opfibration in* **Fib**$(\mathcal{A})$ $\Leftrightarrow$ *S 2-sided fibration*
- (1)+(2 disc.) $\Leftrightarrow$ *S discrete 2-sided fibration (= profunctor)*

Remark

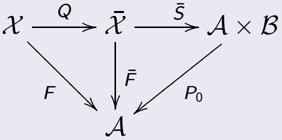
Although the notion of regular span is in principle weaker than that of 2-sided fibration, it is worth observing that the relevant example **Ext**^{n} *is indeed a 2-sided fibration.*

The fibred viewpoint

Next proposition is not present in [Yoneda, 1960], but its proof is!

Proposition

Given a regular span $S\colon \mathcal{X} \to \mathcal{A} \times \mathcal{B}$*, there is a factorization* $S = \bar{S}Q$ *in* $\mathbf{Fib}(\mathcal{A})$*:*



such that, for every object (a, b) *in* $\mathcal{A} \times \mathcal{B}$*,* $\bar{S}_{(a,b)} = \pi_0(S_{(a,b)})$*.*

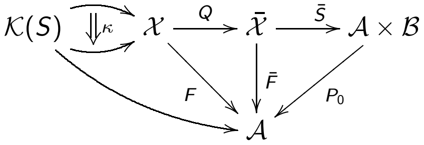
Theorem (CMM, 2017)

Such a factorization is the (initial/discrete opfibration) in $\mathbf{Fib}(\mathcal{A})$*.*

(Internal version of *comprehensive factorization* [Street-Walters, 1973])

The fibred viewpoint

Idea of the proof of the Proposition.



1. The correspondence between $\mathcal{A}$ and $\mathcal{B}$ referred by Yoneda is a profunctor $\bar{S} \colon \mathcal{B} \longrightarrow\!\!\!\!\!\!\!+\ \mathcal{A}$, i.e. a discrete opfibration in $\mathbf{Fib}(\mathcal{A})$.
2. Q is the coidentifier of the identee $(\mathcal{K}(S), \kappa)$ of S in $\mathbf{Cat}$, hence initial and final.
3. Both the identee and the coidentifier live in $\mathbf{Cat}/\mathcal{A}$ and in $\mathbf{Fib}(\mathcal{A})$

Terminology: identee (australian) = kernel cell (french)

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Non abelian settings

Now we want to apply the fibred viewpoint to a fiberwise opfibration such that P_0 is a split fibration:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad P \quad} & \mathcal{M} \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle P_0} & \\ & \mathcal{A} & \end{array}$$

Definition (CMMV, 2016)

This is called *Basic Setting for a Strict Obstruction Theory* (SOT).

Theorem (CMMV, 2016)

In a b.s. for a SOT (P, F, P_0)*, given* x_1, x_2 *in* $\mathcal{X}$*, and* $\mu\colon P(x_1) \to P(x_2)$*, there is a bijection*

$$\mathcal{X}_\mu(x_1, x_2) \xrightarrow{\sim} \mathcal{X}_{P(\mu^*(x_2))}(\mu_*(x_1), \mu^*(x_2))$$

In particular, $\quad \mathcal{X}_\mu(x_1, x_2) \neq \emptyset \quad \Leftrightarrow \quad \mu_*(x_1) \sim \mu^*(x_2)$

Non abelian settings

A leading example we have in mind is

$$\begin{array}{ccc} \mathbf{XExt}^2(\mathbf{Gp}) & \xrightarrow{(\pi_0,\pi_1)} & \mathbf{Mod}(\mathbf{Gp}) \\ & {\scriptstyle \pi_0} \searrow \quad \swarrow {\scriptstyle U} & \\ & \mathbf{Gp} & \end{array}$$

and generalizations. . .

- take crossed n-fold extensions of groups (classical)
- take a (nice) category $\mathcal{C}$, instead of **Gp** (Bourn, Rodelo)

Notice that, since U is split, we have a description

$$\mathbf{Mod}(\mathbf{Gp}) = \mathbf{Gp} \ltimes \coprod_{G \in \mathbf{Gp}} G\text{-}\mathbf{Mod}$$

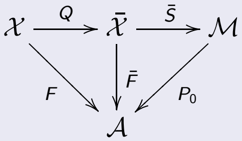
Non abelian settings

Theorem (CMM, 2017)

Given a Basic Setting for a SOT

$$P\colon F \to P_0$$

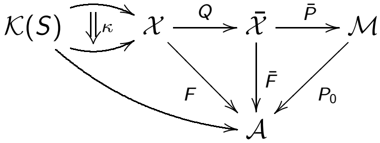
there is a factorization $P = \bar{S}Q$ *in* $\mathbf{Fib}(\mathcal{A})$*:*



such that for every object m *in* $\mathcal{M}$, $\bar{S}_m = \pi_0(S_m)$.
Such a factorization is the (initial/discrete opfibration) in $\mathbf{Fib}(\mathcal{A})$.

Of course, for $\mathcal{M} = \mathcal{A} \times \mathcal{B}$ and P a regular span, we get Yoneda's result.

Non abelian settings



Again, the proof that Q is initial is obtained by showing that it is the coidentifier of $(\mathcal{K}(P), \kappa)$.

On the other hand, we show that $\bar{P}$ is a discrete opfibration in **Fib**$(\mathcal{A})$, but we lose the connection with profunctors.

This fact raises new questions. . .

Breaking the symmetry

Is it possible to give an interpretation of a fiberwise opfibration that generalizes the interpretation of a regular span as a profunctor?

$$\begin{array}{ccc} \bar{\mathcal{X}} & & \bar{\mathcal{X}} \\ \downarrow & \Leftrightarrow & \swarrow \quad \searrow \\ \mathcal{A} \times \mathcal{B} & & \mathcal{A} \qquad\qquad \mathcal{B} \end{array}$$

$$\begin{array}{ccc} \bar{\mathcal{X}} & & \\ \downarrow & \Leftrightarrow & ? \\ \mathcal{A} \rtimes \coprod \mathcal{M}_a & & \end{array}$$

Moreover, so far we analyzed only the functorial properties of $\bar{S}$, but in a (nice) additive setting, regular spans determine also additive structures on the $\bar{S}_{(a,b)}$...

THANK YOU!