Normalities and Commutators

Sandra Mantovani[a], Giuseppe Metere[a]

[a]*Dipartimento di Matematica, Università degli Studi di Milano,*
Via C. Saldini, 50, 20133 Milano, Italia

Abstract

We first compare several algebraic notions of normality, from a categorical viewpoint. Then we introduce an intrinsic description of Higgins' commutator for ideal-determined categories, and we define a new notion of normality in terms of this commutator. Our main result is to extend to any semi-abelian category the following well-known characterization of normal subgroups: a subobject K is normal in A if, and only if, $[A, K] \leq K$.

Key words: normal subobject, ideal, commutator, semi-abelian
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1. Introduction

The notion of *ideal* is central in many algebraic disciplines. As recalled in [GU84, JMU07], in universal algebra ideals were introduced by Higgins [Hig56] for Ω-groups, and by Magari [Mag67] in a more general setting (where not all ideals have to be kernels). Subsequently, Agliano and Ursini [AU92] defined *clots*, a notion lying in-between kernels and ideals.

It was only in 2007 that Janelidze, Márki and Ursini [JMU07] reviewed from a categorical perspective the relationship among normal subobjects, clots and ideals. They described these three concepts in an intrinsic setting, by using the notion of internal action [BJ98, BJK05]. They showed that the relationships

$$N(A) \subseteq C(A) \subseteq I(A)$$

among the sets of normal subobjects, clots and ideals in an object A, become equalities in the semi-abelian case (categorical counterpart of the BIT-speciale varieties of [Urs73]).

Actually, in the varietal case, BIT-speciale axioms are more than enough in order to guarantee $N(A) = I(A)$: this equality holds also in the weaker context of BIT varieties of [Urs72], also called *ideal-determined* in [GU84]. In [JMTU09] the authors showed that the categorical counterpart of BIT varieties

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can be obtained by removing the so-called Hofmann's Axiom from the old-style definition of a semi-abelian category [JMT02]. They called these categories ideal-determined.

In this paper, we first revisit various notions of normalities from kernels to ideals, pointing out the relationships exiting among them in appropriate categorical contexts. Then we move in the ideal-determined case, where all those notions collapse, since the images of kernels along regular epimorphisms are again kernels. This fact allows us to formulate the categorical notion of Higgins' commutator. The last is obtained by taking (through the realization map, see 4.3) the regular image of the *formal commutator*, internal interpretation of the *commutator words* of [Hig56]. By following this approach, we revisit also Huq's commutator [Huq68], showing that in an ideal-determined unital category [BB04], Huq's commutator $[H, K]_Q$ in A is nothing but the normalization in A of Higgins' commutator $[H, K]_H$. The two commutators are different in general, even in the category of groups, if H and K are not normal in A, as Example 4.6 shows.

Nevertheless they always coincide when $H \vee K = A$, in particular if one of the two subobjects is the whole A. In this case we can freely refer to *the* commutator $[A, K]$ (notice that when K is normal, the last coincides also with the normalization of Smith's commutator, as shown in [GVdL08]). The case $H = A$ is special even because in this circumstance the commutator behaves well w.r.t. normalization, in the sense that $[A, K] = [A, \overline{K}]$, as shown in Proposition 4.7.

In the category of groups, the commutator $[A, K]$ can be used to test wether the subgroup K of A is normal in A. Actually K is normal in A if, and only if, $[A, K]$ is a subgroup of K. This characterization of normal subgroups is interesting even because it establish a link between normality and commutators. A natural question is to ask if the internal formulation of this connection is still valid in our settings.

If a category $\mathbb{C}$ is ideal-determined and unital, Proposition 5.1 shows that any normal subobject K of A contains the commutator $[A, K]$. In order to get the converse, we need to use one more ingredient, namely Hoffmann's Axiom, which makes $\mathbb{C}$ into a semi-abelian category. In this context, by means of Proposition 4.7, we can use some results by Bourn and Gran on central extensions [BG02, BG02a, BG02b] proving this way that the full characterization of normality via commutators holds an any semi-abelian category. This result extends what happens for groups to rings, Lie algebras, Leibniz algebras, more generally any variety of ω-groups, as well as to Heyting algebras to the dual category of the category of pointed sets.

2. Normalities

We present some different notions of normality, and try to explain the relationship among them. Although some of those can be defined with few re-

quirements on the base category $\mathbb{C}$, we will assume throughout the paper that $\mathbb{C}$ is (at least) a pointed category with finite limits. Finally, these notions are presented from the strongest to the weakest.

Kernels

This is classical. A map $k : K \to A$ is a *kernel* when there exists a map $f : A \to B$ such that the following is a pullback diagram

$$
\begin{array}{ccc}
K & \longrightarrow & 0 \\
{\scriptstyle k}\downarrow & \lrcorner & \downarrow \\
A & \xrightarrow{\ f\ } & B.
\end{array}
$$

In other words, a kernel is the fiber over the zero-element of the codomain of a certain morphism. Notice that k is a monomorphism, so that kernels are indeed sub-objects.

Normal subobjects

The categorical notion of normality has been introduced for a finitely complete category by Bourn in [Bou00]. Let (R, r_1, r_2) be an equivalence relation on an object A. We call a map $k : K \to A$ *normal* to R when the following two diagrams are pullbacks:

$$
\begin{array}{ccc}
K \times K & \xrightarrow{\ \tilde{k}\ } & R \\
{\scriptstyle id}\downarrow & \lrcorner & \downarrow{\scriptstyle \langle r_1, r_2\rangle} \\
K \times K & \xrightarrow{\ k \times k\ } & A \times A
\end{array}
\qquad
\begin{array}{ccc}
K \times K & \xrightarrow{\ \tilde{k}\ } & R \\
{\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle r_1} \\
K & \xrightarrow{\ k\ } & A.
\end{array}
$$

The map k results to be a mono, and diagrams above express nothing but the fact that the subobject K is an equivalence class of R.

Let us notice that, as $\mathbb{C}$ has a zero object, any equivalence relation determines a unique normal monomorphism: it suffices to take the pullback

$$
\begin{array}{ccc}
K & \longrightarrow & R \\
{\scriptstyle k}\downarrow & \lrcorner & \downarrow{\scriptstyle \langle r_1, r_2\rangle} \\
A & \xrightarrow{\ \langle 1,0\rangle\ } & A \times A,
\end{array}
\tag{1}
$$

i.e. in the pointed case, any normal subobject K is precisely the class $[0]_R$ of an equivalence relation R (see [BB04]).

The kernel of a map f is normal to the usual kernel pair equivalence relation $R[f]$, so that every kernel is normal. The converse is not true in general. It is so when the base category $\mathbb{C}$ is moreover protomodular, with all equivalence relation effective [BB04].

2.1. Seminormal subobjects

P. Agliano and A. Ursini ([AU92]) introduced the notion of *clot* for algebraic varieties.

A subalgebra K of A in $\mathbb{C}$ is a clot in A if

$$t(a_1, \ldots, a_m, 0, \ldots, 0) = 0 \quad \text{and} \quad k_1, \ldots, k_n \in K$$

$$\text{imply} \quad t(a_1, \ldots, a_m, k_1, \ldots, k_n) \in K$$

for any $a_1, \ldots, a_m, k_1, \ldots, k_n$ in A and any $(m+n)$-ary term function t of A.

We will deal with the (weaker) categorical notion of clot in the following section. Here we are interested in a notion introduced [JU09], which is equivalent to that of clot in the varietal case. There, as pointed out in [AU92], a clot is the same as the set (subalgebra) of the elements $x \in A$ such that $0Rx$ for a given internal reflexive relation (i.e. a semicongruence). Hence we call a map $k : K \to A$ *seminormal* in A w.r.t. R when k is obtained by a pullback as in diagram (1), where R is a semicongruence on A (notice that in [JU09] the same is called clot).

Of course every normal subobject is also seminormal, as every equivalence relation is reflective. The converse may not hold in general; however it does hold when the category $\mathbb{C}$ is Mal'cev.

In order to introduce the categorical notion of clot, we recall, for the reader's convenience the notion of *internal action*.

Let $\mathbb{C}$ be a finitely complete pointed category with coproducts. Then for any object B in $\mathbb{C}$ one can define a functor "ker" from the category of split epimorphisms (a.k.a. points) over B into $\mathbb{C}$

$$\ker : Pt_B(\mathbb{C}) \to \mathbb{C}, \qquad \beta \left\uparrow\downarrow\right\alpha \begin{matrix} A \\ \\ B \end{matrix} \; \mapsto \ker(\alpha).$$

This has a left adjoint:

$$\mathrm{B} + (-) : \mathbb{C} \to Pt_B(\mathbb{C}), \quad X \mapsto \; i_B \left\uparrow\downarrow\right [1,0] \begin{matrix} B+X \\ \\ B \end{matrix} ,$$

and the monad corresponding to this adjunction is denoted by $B\flat(-)$. In fact for any object A of $\mathbb{C}$ one gets a kernel diagram:

$$B\flat A \xrightarrow{\;n_{B,A}\;} B + A \xrightarrow{\;[1,0]\;} B.$$

The normal monomorphism $n_{B,A}$ will be denoted simply n when no confusion arises. The $B\flat(-)$-algebras are called internal B-actions in $\mathbb{C}$ (see [BJK05]).

4

Let us observe that in the case of groups, the object $B\flat A$ is the group generated by the *formal conjugates* of elements of A by elements of B, i.e. by the triples of the kind (b, a, b^{-1}) with $b \in B$ and $a \in A$.

For any object A of $\mathbb{C}$, one can define a canonical action of A on A itself given by the composition:

$$\chi_A : \ A\flat A \xrightarrow{n_{A,A}} A + A \xrightarrow{[1,1]} A \ .$$

In the category of groups, the morphism χ_A is the internal actions associated to the usual conjugation in A: the realization morphism $[1,1]$ of above makes the formal conjugates of $A\flat A$ computed effectively in A.

Clots

A subobject $k : K \rightarrowtail A$ is clot in A, when there exists a morphism $\xi : A\flat K \to K$ such that the diagram

$$
\begin{array}{ccc}
A\flat K & \xrightarrow{\ \xi\ } & K \\
{\scriptstyle 1\flat k}\big\downarrow & & \big\downarrow{\scriptstyle k} \\
A\flat A & \xrightarrow[\chi_A]{} & A
\end{array}
$$

commutes. As k is a mono, the morphism ξ defined above is unique: namely it is the internal $A\flat(-)$-action that restricts the action in A.

Every seminormal subobject is closed under conjugation. In fact the restriction map ξ is the internal action on the kernel corresponding to the split extension determined by the codomain map of R.

In [JMU09] it is shown that, when $\mathbb{C}$ is a pointed regular category with finite coproducts, also the converse holds .

Ideals

The categorical notion of ideal has been recently introduced in [JMU09], while the varietal one can be found in [Mag67, Urs72, Hig56]. A subobject $k : K \rightarrowtail A$ is an ideal in A when it is the regular image of a clot along a regular epimorphism, i.e. if there exists a commutative diagram

$$
\begin{array}{ccc}
L & \xdashrightarrow{\ f'\ } & K \\
{\scriptstyle l}\big\downarrow & & \big\downarrow{\scriptstyle k} \\
B & \xdashrightarrow[f]{} & A.
\end{array}
$$

with K clot and f, f' regular epimorphisms. It is immediate to observe that, according to this definition, every clot is an ideal subobject.

The notion of ideal rests on the possibility of taking images, which would lead us to set the definition more appropriately in a regular category. Let us observe that if this is the case, since every clot is the regular image of a kernel, and since in regular categories regular epimorphisms compose, ideals are regular images of kernels (as it happens in relevant algebraic examples, see *Remark 3.3* in [JMU09]). This has an interesting consequence when $\mathbb{C}$ is moreover Mal'cev (see [CLP91]). Bourn has shown [BB04] that in that case, regular images of kernels are normal. Result: ideal and normal subobjects coincide.

The discussion above is summarized in the following table:

KERNEL	NORMAL	SEMINORMAL	CLOT	IDEAL
$\begin{array}{ccc} K & \rightarrow & 0 \\ k\downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$	$\begin{array}{ccc} K & \longrightarrow & R \\ k\downarrow & & \downarrow \\ A & \xrightarrow{\langle 1,0\rangle} & A\times A \\ \downarrow & & \downarrow \pi_2 \\ 0 & \longrightarrow & A \end{array}$ $K=[0]_R,$ R equiv. rel.	$\begin{array}{ccc} K & \longrightarrow & R \\ k\downarrow & & \downarrow \\ A & \xrightarrow{\langle 1,0\rangle} & A\times A \\ \downarrow & & \downarrow \pi_2 \\ 0 & \longrightarrow & A \end{array}$ $K=[0]_R,$ R refl. rel.	$\begin{array}{ccc} A\flat K & \xrightarrow{\xi} & K \\ 1\flat k\downarrow & & \downarrow k \\ A\flat A & \xrightarrow{\chi_A} & A \end{array}$ χ_A factors through k.	$\begin{array}{ccc} L & \xrightarrow{f'} & K \\ l\downarrow & & \downarrow k \\ B & \xrightarrow{f} & A \end{array}$ l clot, f, f' reg. epi

(with implications: $\Longrightarrow$ across all columns; $\Longleftarrow$ under "Mal'cev" between NORMAL and SEMINORMAL; $\Longleftarrow$ under "Regular" between SEMINORMAL and CLOT; $\Longleftarrow$ under "Mal'cev + Regular" between NORMAL and IDEAL)

3. Semi-abelian, homological and ideal-determined categories.

Semi-abelian categories have been introduced in [JMT02] in order to recapture algebraic properties of groups, rings etc. in a categorical-theoretical setting, just as abelian categories do for abelian groups, modules etc.

A semi-abelian category is a Barr-exact, Bourn-protomodular category with zero object and finite coproducts. In [JMT02] the authors compare the definition as stated above (i.e. given in terms of "new" axioms) with preexisting investigations on this subject. This lead them to state equivalent versions of the main definition, given in terms of some so–called "old" axioms, more commonly used in universal algebra. From there we borrow the following characterization: a pointed category $\mathbb{C}$ with finite limits and colimits is semi-abelian if, and only if, it satisfies A1, A2 and A3 below:

A1 $\mathbb{C}$ has a pullback-stable normal-epi/mono factorization

A2 regular images of kernels are kernels

A3 (Hoffmann's axiom) in the diagram below, where l and l' are regular epimorphisms and x' is a kernel, if $\mathrm{Ker}(l) \leq X$ then x is also a kernel.

$$
\begin{array}{ccc}
X & \xrightarrow{\ l'\ } & X' \\
{\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x'} \\
Y & \xrightarrow{\ l\ } & Y'
\end{array}
$$

Axiom A1 can be easily reformulated as follows:

A1$'$ $\mathbb{C}$ is regular and regular epi's = normal epi's.

When axiom A1 holds, axiom A2 means precisely that kernels = ideals, and in this case all the different notions detailed in Section 2 collapse. Pointed categories with finite limits and colimits satisfying axioms A1 and A2 are called *ideal-determined* [JMU09].

Differently, as shown in Proposition 3.3. of [JMT02], axioms A1 and A3 characterize homological categories among those with finite limits and zero object, where homological means pointed, regular and protomodular, according to the definition due to Borceux and Bourn [BB04].

The formulation of the notion of semi-abelian category in terms of the (old) axioms A1, A2 and A3 led us to the following analysis (in the perspective of the observation in 2.7 of [JMT02]).

Let us consider the diagram:
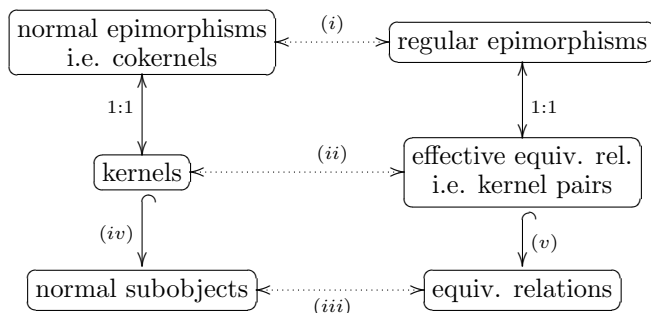
If they are considered separately, the two columns describe quite general facts which hold in categories where referred items do exists: in a category with pullbacks and coequalizers, regular epimorphisms are in one-to-one correspondence with kernel pairs, and any kernel pair is an equivalence relation. Similarly, in a category with kernels and cokernels one gets a one-to-one correspondence between them, and any kernel is a normal subobject.

Now, axiom A1, in the form of A1$'$, amounts to $\mathbb{C}$ being a regular category, plus the arrow (i) being an equality. This implies immediately that the unique map (ii), that makes the upper square commute, is a bijection, and hence the following valuable fact holds: any effective equivalence relation is completely determined by its zero-class.

When also axiom A3 holds, i.e. when the category $\mathbb{C}$ is homological, this fact can be extended to equivalence relations, that are univocally determined by their zero classes [BB04]. In this case, the bijection (ii) is just the restriction of a more general bijection (iii).

Finally let us consider the new axiom:

A2′ Kernels = normal subobjects.

This means precisely that the inclusion (iv) is indeed an equality. Hence, when the category $\mathbb{C}$ is homological, this fact collapses the lower square of the diagram, so that axiom A2′ is equivalent to the inclusion (v) being an equality. This means that the regular category $\mathbb{C}$ is moreover exact.

Considerations above give the following characterization:

Corollary 3.1. *A category $\mathbb{C}$ is semi-abelian if, and only if, axioms A1′, A2′ and A3 hold.*

4. Commutators

Several notions of commutators of two coterminal morphisms (i.e. with the same codomain) have been proposed and studied in different algebraic contexts.

In this section we explore some aspects of commutator theory, by comparing categorically the two different notions of commutator introduced by Higgins [Hig56] and Huq [Huq68]. Our approach rests on the observation that both these notions can be formulated from the same notion of *formal commutator*, but first we need to recall the original definitions.

4.1. Higgins' commutator

The notion of commutator according to Higgins, generalizes the case of groups to those of Ω-groups [Hig56].

We recall the definition of Ω-groups for the reader's convenience. A category $\mathcal{V}$ of Ω-groups is a variety of groups (in the sense of the universal algebra) such that:

- the group identity is the only operation of arity 0, i.e. the variety is pointed;

- all other operations different from group operation (here written additively), inverse and identity, have arity n, with $n \geq 1$.

Given a Ω-group A and two subobjects $h : H \rightarrowtail A$ and $k : K \rightarrowtail A$, the Higgins' commutator $[H, K]_H$ is set of all $f(\overrightarrow{h}, \overrightarrow{k})$ with $\overrightarrow{h} \in H^n$ and $\overrightarrow{k} \in K^m$, f being commutator words, i.e. such that $f(\overrightarrow{0}, \overrightarrow{y}) = 0 = f(\overrightarrow{x}, \overrightarrow{0})$.

Actually it is possible to describe Higgins' commutator as the image of a certain subobject of the coproduct $H+K$ through the canonical map $[h,k] : H + K \to A$. This can be given in an intrinsic way, which will make a generalization easier.

Let us fix some notation. Let H and K be two given objects of a pointed finitely complete category $\mathbb{C}$ with coproducts. We denote by $\Sigma_{H,K}$ (or simply Σ) the canonical arrow

$$\Sigma_{H,K} = \langle [1,0], [01] \rangle = [\langle 1,0 \rangle, \langle 0,1 \rangle] : H + K \to H \times K.$$

Let us observe that $\Sigma_{H,K}$ is a regular epimorphism for every pair of objects H and K in $\mathbb{C}$ if, and only if, $\mathbb{C}$ is unital [BB04].

4.2. Formal commutator

Let us consider two objects H and K of $\mathbb{C}$. We define the *formal commutator* of H and K as the kernel $(H \diamond K, \sigma_{H,K})$ of $\Sigma_{H,K}$. Actually $H \diamond K$ is the intersection of $H \flat K$ and $K \flat K$, as shown in the diagram below:

$$
\begin{array}{ccccc}
H \diamond K & \longrightarrow & H \flat K & & \\
\downarrow & \overset{\sigma_{H,K}}{\searrow} & \downarrow & & \\
K \flat H & \longrightarrow & H + K & \xrightarrow{[0,1]} & K \\
& {\scriptstyle [1,0]}\downarrow & \overset{\Sigma_{H,K}}{\searrow} & & \uparrow \\
& & H & \longleftarrow & H \times K.
\end{array}
$$

The reason why we call $H \diamond K$ *formal commutator* is, from one side, that it is just the Huq's commutator (that we are going to define later) of H and K in $H + K$ (see Remark 2).

Our motivation for using the therm *formal* originates by the fact that, in the category of groups, the elements of $H + K$ can be represented as reduced formal juxtapositions of elements of H and K, say sequences of the kind $(h_1, k_1, \cdots, h_n, k_n)$. $H \diamond K$ is generated by all the words of the kind (x, y, x^{-1}, y^{-1}), with $x \in H$ and $y \in K$.

4.3. The realization map

The canonical map

$$[h,k] : H + K \to A$$

has an interesting interpretation when the object A is a group (actually, similar arguments hold in any pointed variety of universal algebra). The map $[h,k]$ acts on sequences of the kind $(h_1, k_1, \cdots, h_n, k_n)$ by means of the group operation of A, thus giving the element $h_1 k_1 \cdots h_n k_n$ *computed* in A.

Now, if (H,h) and (K,k) are subgroups of a given group A, one can easily check that the image of $H \diamond K$ through $[h,k]$ is precisely the commutator subgroup of H and K in A.

At this point one can easily define the internal version of Higgins' commutator in an ideal-determined category.

Definition 4.1. *Let $\mathbb{C}$ be ideal-determined. Then the Higgins' commutator $[H, K]_H$ of two subobjects $h : H \rightarrowtail A$ and $k : K \rightarrowtail A$ of A is the regular image of $H \diamond K$ under the morphism $[h, k]\sigma_{H,K}$ (see diagram below)*

$$
\begin{array}{ccc}
H \diamond K & \twoheadrightarrow & [H, K]_H \\
\scriptstyle{\delta_{H,K}} \big\downarrow & & \big\downarrow \\
H + K & \xrightarrow[{[h,k]}]{} & A.
\end{array}
$$

As far as the ground category $\mathbb{C}$ has a regular epi/mono factorization system, as in the ideal-determined case, we can obtain the join of two subobjects (H, h) and (K, k) of a given object A as the regular image of the canonical map $[h, k]$:

$$
\begin{array}{ccc}
& H \vee K & \\
\scriptstyle{p} \nearrow & & \searrow \scriptstyle{q} \\
H + K & \xrightarrow[{[h,k]}]{} & A.
\end{array}
$$

Then, the following proposition holds.

Proposition 4.2. *Let $\mathbb{C}$ be ideal-determined. Then the Higgins' commutator $[H, K]_H$ of two subobjects $h : H \rightarrowtail A$ and $k : K \rightarrowtail A$ of A is a normal subobject of $H \vee K$.*

PROOF. The inclusion of $[H, K]_H$ in A factors through $H \vee K$ by a monomorphism, which is normal as a consequence of axiom A2; see diagram below:

$$
\begin{array}{ccc}
H \diamond K & \twoheadrightarrow & [H, K]_H \\
\scriptstyle{\delta_{H,K}} \big\downarrow & & \big\downarrow \searrow \\
H + K & \twoheadrightarrow H \vee K \rightarrowtail & A.
\end{array}
$$
$$[h,k]$$

4.4. Huq's commutator

The construction described below was introduced by S. A. Huq [Huq68] in a purely categorical setting, and further developed by D. Bourn [Bou04]. We borrow the general definition from the second author, but we will restrict our attention to the cases when the two coterminal morphisms are monomorphisms.

Definition 4.3. *Let $\mathbb{C}$ be a finitely complete, unital category, such that regular and normal epimorphisms coincide, and let us consider two subobjects $h \colon H \rightarrowtail$*

A and $k : K \rightarrowtail A$. We define the commutator quotient $Q = Q(h,k)$ as the colimit of the solid arrows in the diagram below, if it exists:

$$(2)$$

The kernel of q is denoted by $[H,K]_Q$ and we will refer to it as to the Huq's commutator of H and K.

Let us observe that when the map q is an isomorphism, this means precisely that h and k cooperate, i.e. H and K commute in A. Hence the distance for q from being an isomorphism expresses exactly the lack of commutativity between H and K. Since q is a normal epimorphism, this can be measured by its kernel $[H,K]_Q$.

Remark1. All that follows is indeed well known, but it is worth recalling it, as it describes the commutator (quotient) as a kind of universal represener of the algebraic operations.

Let $\mathbb{C}$ be the category of groups. The group operation $\cdot : H \times K \dashrightarrow A$ is not a morphism in general. Nevertheless one can quotient the group A with some normal subgroup, such that the quotient map is indeed a morphism. This is always possible: a trivial answer is to quotient A with itself. A better answer is given by the commutator of H and K. This is in fact the smaller normal subgroup such that such that the quotient map is a morphism: with $Q = A/[H,K]_Q$ one has:

$$H \times K \xrightarrow{\;\;\dot{-}\;\;}_{\displaystyle m} A \xrightarrow{\;q\;} A/[H,K]_Q.$$

Finally, if $H = K = A$ we get the Heckmann-Hilton argument: a group A is abelian if, and only if, the group operation $A \times A \to A$ is a morphism, and this happens precisely when the derived subgroup $[A,A]_Q$ is trivial.

We observe that the definition of the commutator quotient can be expressed by means of a canonical pushout, as one can easily see in the next proposition:

Proposition 4.4. *Let $\mathbb{C}$ be as before, with finite sums. The colimit diagram*

(2) is equivalent to the following pushout

$$
\begin{array}{ccc}
H + K & \xrightarrow{[h,k]} & A \\
{\scriptstyle \Sigma}\downarrow & & \downarrow{\scriptstyle q} \\
H \times K & \xrightarrow{\ \ m\ \ } & Q
\end{array}
\qquad . \tag{3}
$$

In this context, the Huq's commutator $[H, K]_Q$ of H and K is obtained as the kernel of the map q of diagram (3).

Remark2. It is worth considering the very special case when $A = H + K$, and h and k are the canonical inclusions. In this situation diagram (3) trivializes, and the Huq's commutator $[H, K]_Q$ of H and K in $H + K$ coincides with the formal commutator $H \diamond K$, whence its name.

4.5. Higgins commutator and Huq commutator compared

The discussion above suggests to develop some considerations. Let $\mathbb{C}$ be an ideal-determined, unital category and let H, K be subobjects of an object A as above. This is a nice context where Huq's commutator and Higgins' commutator can be compared.

In general these two notions do not coincide, more precisely $[H, K]_H$ is a proper subobject of $[H, K]_Q$, since q restricted to $[H, K]_H$ is zero, as the following diagram shows:

Let us observe that in the above diagram, the three vertical sequence of morphisms are exact and the two bottom "diamonds" are pushouts.

To be more precise, the relationship between Huq's commutator and Higgins' commutator is explained by the following proposition.

Proposition 4.5. *Let* $\mathbb{C}$ *be an ideal-determined, unital category, and let* H *and* K *be subobjects of* A. *Then*

$$[H,K]_Q = \overline{[H,K]}_H,$$

i.e. the Huq's commutator is the normalization in A *of the Higgins' commutator.*

PROOF. Since $\mathbb{C}$ is unital, Σ is a regular epi, and so is q. Actually both are cokernels by axiom A1, and a direct calculation shows that q is precisely the cokernel of the inclusion $[H,K]_H \rightarrowtail A$.

A natural question to ask at this point is under what conditions on $\mathbb{C}$ the comparison is an isomorphism, so that the two notions coincide. As we already noticed, this is not true in general, and even for the category of groups the two notions are distinct, if H and K are not normal, as the following example shows.

Example 4.6. *Let us consider the simple group* A_5, *given by even permutations of order five, and the two subgroups* $H = \langle(12)(34)\rangle$ *and* $K = \langle(12)(45)\rangle$. *Then* $[H,K]_H = \langle(345)\rangle \neq [H,K]_Q = A_5$ *(see [Cig10] for a detailed discussion).*

Of course, Huq's commutator and Higgins' commutator coincide when the subobjects are sufficiently big, i.e. when $H \vee K = A$ (as, for example, in the case of the formal commutator, where $H \vee K = H + K$). In particular, this happens when one of the subobjects, say H, is the whole A; then the map $[1,k] : A + K \to A$ is a regular epimorphism, and the diagram above happily collapses. If this is the case, we will drop the H and the Q subscripts, and write simply $[A,K]$ for *the* commutator of A and K. Another reason why the case $H = A$ is special is that in this case Huq's commutator behaves well w.r.t. normalization, as shown in the Proposition 4.7 below. In general this is not true, as one can verify for the simple group A_6, with $H = \langle(123)\rangle$ and $K = \langle(456)\rangle$, where $[H,K]_Q = 0$, while $[\overline{H},\overline{K}]_Q = A_6$ (see [Cig10]).

Proposition 4.7. *Let* $\mathbb{C}$ *be an ideal-determined unital category. Then for a monomorphism* $k : K \rightarrowtail A$ *one has*

$$[A,K] = [A,\overline{K}],$$

where $(\overline{K}, \bar{k})$ *is the normalization of* (K,k).

PROOF. Let us refer to the two diagrams below:

$$
\begin{array}{ccccc}
A \diamond (A\flat K) & \xrightarrow{\;q\;} & A \diamond K & \xrightarrow{\;p\;} & [A,K] \\
{\scriptstyle D}\big\downarrow & (i) & {\scriptstyle d}\big\downarrow & (ii) & {\scriptstyle c}\big\downarrow \\
A\flat(A\flat K) & \xrightarrow[(\chi_A\flat A)\varphi]{} & A\flat K & \xrightarrow[\;\chi\;]{} & \overline{K},
\end{array}
$$

$$A \diamond (A\flat K) \xrightarrow{\ \overline{q}\ } A \diamond \overline{K} \xrightarrow{\ \overline{p}\ } [A, \overline{K}]$$

$$\left\downarrow D \quad (iii) \quad \left\downarrow \overline{d} \quad (iv) \quad \left\downarrow \overline{c}\right.$$

$$A\flat(A\flat K) \xrightarrow{\ A\flat\chi\ } A\flat\overline{K} \xrightarrow{\ \overline{\chi}\ } \overline{K}.$$

The idea of the proof is to show that the compositions pq and $\overline{p}\overline{q}$ are the regular images of the same (to be proved) morphism $\chi(\chi_A\flat K)\varphi D = \overline{\chi}(A\flat\chi)D$. We shall start by defining all the characters playing in the diagrams.

- The arrows D, d and $\overline{d}$ are kernels:

$$A \diamond (A\flat K) \rightarrowtail^{D} A\flat(A\flat K) \xrightarrow{[0,1]n} A\flat K$$

$$A \diamond K \rightarrowtail^{d} A\flat K \xrightarrow{[0,1]n} K, \qquad A \diamond \overline{K} \rightarrowtail^{\overline{d}} A\flat\overline{K} \xrightarrow{[0,1]n} \overline{K}.$$

- The arrows χ and $\overline{\chi}$ are the regular images of the compositions:

$$A\flat K \rightarrowtail^{n} A + K \xrightarrow{[1,k]} K, \qquad A\flat\overline{K} \rightarrowtail^{n} A + \overline{K} \xrightarrow{[1,k]} \overline{K}.$$

- In diagrams (ii) and (iv), up-right compositions are regular epimorphisms followed by monomorphisms, factorizing left-down compositions.

- The arrow φ is given by universal property of kernels, see the diagram below, where the lower part commutes:

$$A\flat(A\flat K) \cdots\cdots\cdots\cdots\xrightarrow{\ \varphi\ }\cdots\cdots\cdots\cdots (A\flat A)\flat K$$

$$\left\downarrow n \qquad\qquad\qquad\qquad\qquad\qquad \left\downarrow n\right.$$

$$A + (A\flat K) \xrightarrow{A+n} A + A + K \xrightarrow{[\eta_A,\eta_A]+K} (A\flat A) + K$$

$$\left\downarrow [1,0] \qquad\qquad\qquad\qquad\qquad\qquad \left\downarrow n[1,0]\right.$$

$$A \xrightarrow{\qquad\qquad \eta_A \qquad\qquad} A\flat A.$$

In fact:
$$[1,0]([\eta_A,\eta_A]+K)(A+n) =$$
$$= [1,0](\eta_A K)([1,1]+K)(A+n) = \eta_A[1,0]([1,1]+K)(A+n) =$$
$$= \eta_A[1,1](A+[1,0])(A+n) = \eta_A[1,1]i_1[1,0] = \eta_A[1,0].$$

- The arrow q is given again by universal property of kernels, so that diagram (i) commutes for free:

$$
\begin{array}{ccccc}
A \diamond (A\flat K) & \xdashrightarrow{\;\;\;\;\;\;\;\;\; q \;\;\;\;\;\;\;\;\;} & & & A \diamond K \\
\downarrow{\scriptstyle D} & & & & \downarrow{\scriptstyle d} \\
A\flat(A\flat K) & \xrightarrow{\;\;\varphi\;\;} & (A\flat A)\flat K & \xrightarrow{\;\chi_A \flat K\;} & (A\flat K) \\
\downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} \\
A + (A\flat K) & \xrightarrow{A+n} A+A+K \xrightarrow{[\eta_A,\eta_A]+K} & (A\flat A) + K & \xrightarrow{\chi_A + K} & (A+K) \\
\downarrow{\scriptstyle [0,1]} & \downarrow{\scriptstyle [0,1_{A+K}]} & \downarrow{\scriptstyle [0,0,1]} & & \downarrow{\scriptstyle [0,1]} \\
A\flat K & \xrightarrow{n} A+K \xrightarrow{[0,1]} & K & \xrightarrow{Id} & K.
\end{array}
$$

In fact:

$$[1,0]n(\chi_A\flat K)\varphi = [1,0](\chi_A+K)n\varphi = [1,0](\chi_A+K)([\eta_A,\eta_A]+K)(A+n)n =$$

$$= [0,0,1](n+K)([\eta_A,\eta_A]+K)(A+n)n = [0,0,1]([i_2,i_2]+K)(A+n)n =$$

$$= [0,0,1](A+n)n = [0,1]n[0,1]n.$$

Moreover q is a split epimorphism, and hence regular. This can be shown by precomposing the upper part of the diagram above with the monomorphism $A \diamond \eta_K$. The calculation shows that $ndq(A \diamond \eta_K) = nd$, which is a monomorphism. Canceling nd, one gets $q(A \diamond \eta_K) = 1_{A\diamond K}$.

- The arrow $\bar{q}$ is simply $A \diamond \chi$, so that diagram (iii) commutes. Moreover $\bar{q}$ is a regular epimorphism by Lemma 4.8.

The discussion above has shown that diagrams (i), (ii), (iii) and (iv) commute, and that their upper sides are regular epimorphisms. Moreover they are followed by monomorphisms. Now we are to show that their lower sides coincide: uniqueness of the factorization will conclude the proof.

Indeed, it is convenient to compose the morphisms that we want to prove equal, with the normal monomorphism $\bar{k}$. On one side one has: $\bar{k}\overline{\chi}A\flat\chi = [1,\bar{k}](A+\chi)n$. The other can be represented by the commutativity of the

diagrams below:

$$
\begin{array}{ccccc}
A\flat(A\flat K) & \xrightarrow{\ \ (\chi_A\flat K)\varphi\ \ } & A\flat K & \xrightarrow{\ \chi\ } & \overline{K} \\
{\scriptstyle n}\downarrow & & {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle \bar{k}} \\
A+(A\flat K) & \xrightarrow{A+n} A+A+K \xrightarrow{[1,1]+k} A+K \xrightarrow{[1,k]} & & & A
\end{array}
$$

with $A+[1,K]$, $A+A$, $A+\chi$, $A+\bar{k}$, $A+\overline{K}$, $[1,1]$, $1,\bar{k}$.

Lemma 4.8. *Let the category $\mathbb{C}$ be ideal-determined, and let X be an object of $\mathbb{C}$. Then*

- *the functors $(-)\flat X$ and $X\flat(-)$ preserves the regular epimorphisms.*

- *the functors $(-)\diamond X$ and $X\diamond(-)$ preserves the regular epimorphisms.*

PROOF. Let us consider the following two diagrams:

$$
\begin{array}{ccc}
X\flat A & \xdashrightarrow{\ m\ } & Z \\
{\scriptstyle n}\downarrow & (i) & \downarrow \\
X+A & \xrightarrow{X+f} & X+B \\
{\scriptstyle [1,0]}\downarrow & (ii) & \downarrow{\scriptstyle [1,0]} \\
X & \xrightarrow{\ 1\ } & X
\end{array}
\qquad
\begin{array}{ccc}
A\flat X & \xdashrightarrow{\ m'\ } & Z' \\
{\scriptstyle n}\downarrow & (iii) & \downarrow \\
A+X & \xrightarrow{f+X} & B+X \\
{\scriptstyle [1,0]}\downarrow & (iv) & \downarrow{\scriptstyle [1,0]} \\
A & \xrightarrow{\ f\ } & B
\end{array}
$$

The square diagrams (ii) and (iv) are pushout: the first because the arrow $f+X$ is a regular epimorphism since f and 1_X are, the second for the fact that the outer and the left-most rectangles below are cokernels:

$$
\begin{array}{ccccc}
X & \xrightarrow{\ i_X\ } & A+X & \xrightarrow{\ f+X\ } & B+X \\
\downarrow & & {\scriptstyle [1,0]}\downarrow & & \downarrow{\scriptstyle [1,0]} \\
0 & \xrightarrow{\ \ \ } & A & \xrightarrow{\ f\ } & B.
\end{array}
$$

Hence one can take regular images (i) and (iii); since the category is ideal-determined, they will coincide with the kernels of the pushout:

$$
Z = X\flat B, \quad m = X\flat f; \qquad Z' = B\flat X, \quad m' = X\flat f.
$$

This completes the proof of the first part of the lemma.

In order to prove the second part of the lemma, we will point our attention to the functor $(-) \diamond X$, as the two are naturally isomorphic.

The argument of the proof is similar to that of the first part. Let us consider the diagram:

$$
\begin{array}{ccc}
X \diamond A & \xrightarrow{\ \ m\ \ } & Z \\
\downarrow{\scriptstyle d} & & \downarrow \\
X\flat A & \xrightarrow{\ X\flat f\ } & X\flat B \\
\downarrow{\scriptstyle n} \ \ {\scriptstyle (v)} & & \downarrow{\scriptstyle n} \\
X + A & \xrightarrow{\ X+f\ } & X + B \\
\downarrow{\scriptstyle [0,1]} \ \ {\scriptstyle (vi)} & & \downarrow{\scriptstyle [0,1]} \\
A & \xrightarrow{\ \ f\ \ } & B.
\end{array}
$$

Here, again, (m, Z) is the regular image of $(X\flat f)n$, so that it suffices to show that $(v) + (vi)$ is a pushout in order to get the result. The proof that (iv) is a pushout is essentially the same we saw for (iv), while for (v) some more calculations are needed.

Let two converging arrow α and β be given, such that $\beta n = \alpha(X\flat f)$. Then, by universal property of the coproduct $X + B$ we get a unique γ such that $\alpha\eta_B = \gamma n\eta_B$ and $\beta i_X = \gamma(X + f)i_X$, where $i_X : X \to X + A$ is the coproduct injection.

Claim: $\beta = \gamma(X + f)$ and $\alpha = \gamma n$. The first claim is obtained by precomposing with coproduct injections into $A + X$. The first one, $\beta i_X = \gamma(X + f)i_X$, is given above. For the second one, just follow the chain of equalities:

$$\beta i_a = \beta n\eta_A = \alpha(X\flat f)\eta_A = \alpha\eta_B f = \gamma n\eta_B f = \gamma(X + f)n\eta_A = \gamma(X + f)i_A,$$

where the second equality holds by hypothesis, the fourth is given above and the other by definition. Finally we have to prove the second claim, but it is a consequence of the first one, which justifies the middle equality in the chain below:

$$\gamma(X\flat f) = \gamma(X + f)n = \beta n = \alpha(X\flat f).$$

5. (Yet) another notion of normality

In the category of groups, one can characterize normal subgroups in many different equivalent ways. The one we present below is surely not one of the best known. Nevertheless it has interesting implications: it establishes a link between the notion of normal subgroup with that of commutator. Moreover it extends unexpectedly to the semi-abelian setting.

5.1. The case of groups

Let K be a subgroup of a given (multiplicative) group A. Then K is normal in G if, and only if, $[A, K] \leq K$. In fact, if K is normal in A, for every pair of elements $k \in K$ and $a \in A$, $ak^{-1}a^{-1} \in K$. Hence also $kak^{-1}a^{-1} \in K$, i.e. all generators of $[A, K]$ are in K. Conversely, whenever $[A, K] \leq K$, for every pair of elements $k \in K$ and $a \in A$ one has $y = k^{-1}aka^{-1} \in K$, so that $ky = aka^{-1} \in K$, i.e. K is normal in A.

One implication of the characterization given above holds in a quite general setting. This is established by the following:

Proposition 5.1. *Let $\mathbb{C}$ be an ideal-determined unital category. If K is a normal subobject of A, then $[A, K]$ is a subobject of K.*

PROOF. Let us refer to the diagram below:

Firstly we consider the pushout of $[1, 0]$ along $[1, k]$. Precomposition with the injection $A \rightarrowtail A + K$ forces the other two morphisms of the pushout to be equal. Let's call them $p\colon A \to C$.

We claim that (C, p) is the cokernel of k. Actually, if we denote by i_K the canonical inclusion of K into $A + K$, one can compute $pk = p[1, k]i_K = p[1, 0]i_K = p0 = 0$. Moreover, for any other morphism such that $fk = 0$, one easily shows that $f[1, k] = f[1, 0]$. Universality of pushouts gives the morphism from C stating that $C = A/K$.

But k is a kernel, so that $k = ker(p)$. Hence, in order to prove that $[K, A] \leq K$ it suffices to show that $pj = 0$. This is done by factoring p by $q = coker(j)$, since the square ($\bullet$) is a pushout by definition.

In order to get the full characterization of a normal subobject in terms of its commutator subobject, we shall move to the semi-abelian setting. This is done in Theorem 5.3 below, but first we need to present the following quite general:

Lemma 5.2. *Let $\mathbb{C}$ be a unital category. Given a morphism $\alpha : A \times K \to A \times N$ such that*

$$\pi_A \alpha = \pi_A, \qquad \alpha\langle 1, 0\rangle = \langle 1, 0\rangle,$$

18

then $\alpha = 1 \times r$, where $r = \pi_N \alpha \langle 0, 1 \rangle$. Furthermore, r is a regular epimorphism, if α is.

PROOF. Since $\mathbb{C}$ is unital, $\alpha = 1 \times r$ if and only if the equality holds when precomposing with the canonical injections into the product, that is

$$(i) \quad \alpha \langle 1, 0 \rangle = (1 \times r) \langle 1, 0 \rangle,$$

$$(ii) \quad \alpha \langle 0, 1 \rangle = (1 \times r) \langle 0, 1 \rangle.$$

Since α and $1 \times r$ are morphisms to a product, it is sufficient to test the equalities above when composing with the projections. But

$$(i) \ \pi_A \alpha \langle 1, 0 \rangle = \pi_A \langle 1, 0 \rangle = 1 = \pi_A (1 \times r) \langle 1, 0 \rangle,$$

$$\pi_N \alpha \langle 1, 0 \rangle = \pi_N \langle 1, 0 \rangle = 0 = \pi_N (1 \times r) \langle 1, 0 \rangle$$

and

$$(ii) \ \pi_A \alpha \langle 0, 1 \rangle = \pi_A \langle 0, 1 \rangle = 0 = \pi_A (1 \times r) \langle 1, 0 \rangle, 4$$

$$\pi_N \alpha \langle 1, 0 \rangle = r = \pi_N (1 \times r) \langle 1, 0 \rangle.$$

Consequently $\alpha = 1 \times r$. Furthermore, $\pi_N \alpha = \pi_N (1 \times r) = r \pi_K$, hence, when α is a regular epimorphism, so is r.

Theorem 5.3. *Let $\mathbb{C}$ be a semi-abelian category, and $K \rightarrowtail A$. Then K is a normal subobject of A if, and only if, $[A, K]$ is a subobject of K.*

PROOF. Since any semi-abelian category is both ideal-determined and unital, the necessary condition is given by the Proposition 5.1. So we have to prove that, given $[A, K] \leq K \leq A$, K is normal in A.

If $p : A \rightarrow C$ is the cokernel of k and $\overline{k} : \overline{K} \rightarrow A$ is the kernel of p, then $p = \text{coker}(\overline{k})$. As we have seen in the Proposition 5.1, p can be factorized as $p = \psi q$, where $q : A \rightarrow B$ is the cokernel of $j : [A, K] = [A, \overline{K}] \rightarrow A$ and $\psi : B \rightarrow C$ is given by the universal property of cokernels. Since $\overline{K}$ is normal in A, according to Theorem 2.8.11 of [BB04] we can say that $\psi : B = A/[A, \overline{K}] \rightarrow C = A/\overline{K}$ is central, that is $[B, N] = 0$, where $n : N \rightarrow B$ denotes the kernel of ψ. In other words, this means that there exists a cooperator $\theta : B \times N \rightarrow B$, such that

$$\theta \langle 0, 1 \rangle = n \qquad \text{and} \qquad \theta \langle 1, 0 \rangle = 1_B$$

or, equivalentely, that the diagram

$$
\begin{array}{ccc}
B \times N & \xrightarrow{\theta} & B \\
\pi_B \downarrow & & \downarrow \psi \\
B & \xrightarrow{\psi} & C
\end{array}
\quad .
$$

is a pullback diagram. Then, in the following diagram

$$
\begin{array}{ccccc}
A \times N & \xrightarrow{q \times 1} & B \times N & \xrightarrow{\theta} & B \\
\pi_A \downarrow & & \pi_B \downarrow & & \downarrow \psi \\
A & \xrightarrow{q} & B & \xrightarrow{\psi} & C
\end{array} \quad .
$$

the outer rectangle is a pullback of $p = \psi q$ along ψ. But,

$$
\begin{array}{ccc}
A + K & \xrightarrow{[1,k]} & A \\
& \downarrow \Sigma & \downarrow q \\
[1,0]\ \ A \times K & \xrightarrow{\varphi} & B \ \ p \\
& \downarrow \pi_A & \downarrow \psi \\
A & \xrightarrow{p} & C.
\end{array}
$$

since the outer and the top rectangles in the above diagram are pushouts, the bottom one is a pushout of regular epimorphisms and then a regular pushout, as it happens in any semi-abelian category. This means that the morphism $\alpha : A \times K \to A \times N$ given by the universal property of the pullback:

is a regular epimorphism. Furthermore, $\theta(q \times 1)\alpha = \varphi$ and $\alpha \pi_A = \pi_A$. In order to apply Lemma 5.2, we need only to show that $\alpha \langle 1, 0 \rangle = \langle 1, 0 \rangle$. Since are both morphisms to a pullback, it is sufficient to show that the equality holds when composing with the projections.

$$
(\pi_A \alpha)\langle 1, 0 \rangle = \pi_A \langle 1, 0 \rangle = 1_A = \pi_A \langle 1, 0 \rangle_N
$$

$$
\theta(q \times 1)\alpha\langle 1, 0 \rangle = \varphi\langle 1, 0 \rangle = \varphi \Sigma i_A = q[1, k]i_A = q = \theta\langle 1, 0 \rangle q = \theta(q \times 1)\langle 1, 0 \rangle
$$

the last equalities holding, by the properties of the cooperator θ. Applying Lemma 5.2, we get $\alpha = 1 \times r$, with r regular epimorphism, since α is. Hence $(q \times 1)\alpha = (q \times 1)(1 \times r) = (q \times r)$ and then the following diagram commutes:

$$K \xrightarrow{\langle 0,1 \rangle} A \times K$$

Since the two bottom squares are pullbacks, $\theta \langle 0,1 \rangle = n$. Furthermore, $\varphi \langle 0,1 \rangle = \varphi \Sigma i_K = q[1,k]i_K = qk$; in conclusion we have the following commutative diagram:

$$K \xrightarrow{r} N$$

where the horizontal arrows are regular epimorphisms, the rightmost one is a normal monomorphism and the leftmost one is a monomorphism. But $\mathrm{Ker}(q) = [A,K] \leq K$ by hypothesis, so we can apply Hoffmann's axiom and concude that K is normal in A.

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