PROFUNCTORS IN MALTCEV CATEGORIES AND FRACTIONS OF FUNCTORS

S. MANTOVANI, G. METERE, E. M. VITALE

Abstract. We study internal profunctors and their normalization under various conditions on the base category. In the Maltcev case we give an easy characterization of profunctors. Moreover, when the base category is efficiently regular, we characterize those profunctors which are fractions of internal functors with respect to weak equivalences.

1. Introduction

In this paper we study internal profunctors in a category $\mathcal{C}$, from diverse points of view. We start by analyzing their very definition in the case $\mathcal{C}$ is Maltcev.
It is well known that the algebraic constraints inherited by the Maltcev condition make internal categorical constructions often easier to deal with. For instance, internal categories are groupoids, and their morphisms are just morphisms of the underlying reflective graphs. Along these lines, internal profunctors, that generalize internal functors are involved in a similar phenomenon: it turns out (Proposition 3.1) that in Maltcev categories one of the axioms defining internal profunctors can be obtained by the others.

Our interest in studying internal profunctors in this context, comes from an attempt to describe internally monoidal functors of 2-groups as weak morphisms of internal groupoids. In the case of groups, the problem has been solved by introducing the notion of butterfly by B. Noohi in [22] (see also [2] for a stack version). In [1] an internal version of butterflies has been defined in a semi-abelian context, and it has been proved that they give rise to the bicategory of fractions of $\mathbf{Grpd}(\mathcal{C})$ (the 2-category of internal groupoids and internal functors) with respect to weak equivalences.
In [1] it is observed also that, through a denormalization process, butterflies in a semi-abelian category correspond to *fractors* (Definition 5.2). They give rise to a special kind of internal profunctors, that have been independently considered by D. Bourn in [11] under the name of *regularly fully faithful profunctors*.
Along these lines, by Proposition 4.6 it turns out that in a semi-abelian context, internal profunctors can be represented by internal *crossed profunctors*, whose name was adopted by M. Jibladze in [17] for the case of groups.
Furthermore, the biequivalence

$$\mathbf{XProf}(\mathcal{C}) \simeq \mathbf{Prof}(\mathcal{C})$$

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between internal crossed profunctors and profunctors restricts to a biequivalence

$$\mathbf{Bfly}(\mathcal{C}) \simeq \mathbf{Fract}(\mathcal{C})$$

between butterflies and fractors.

Having these facts in mind, a natural question arise: using fractors instead of butterflies, is it possible to describe the bicategory of fractions of $\mathbf{Grpd}(\mathcal{C})$ with respect to weak equivalences even if $\mathcal{C}$ is no longer semi-abelian?

In this paper we give a positive answer to this question, by proving that fractors are the bicategory of fractions of $\mathbf{Grpd}(\mathcal{C})$ provided that $\mathcal{C}$ is efficiently regular (see [11]), and *a fortiori* when it is Barr-exact (Theorem 6.2).

We can resume the situation with the following picture, where the solid part is what is already known and the dashed part is what we prove in this paper

$$\begin{array}{ccccc}
\mathbf{Grpd}(\mathcal{C}) & \xrightarrow{\mathcal{F}} & \mathbf{Fract}(\mathcal{C}) & \hookrightarrow & \mathbf{Prof}(\mathcal{C}) \\
\simeq \big| & & \big| \simeq & & \big| \simeq \\
\mathbf{XMod}(\mathcal{C}) & \xrightarrow{\mathcal{F}} & \mathbf{Bfly}(\mathcal{C}) & \hookrightarrow & \mathbf{XProf}(\mathcal{C})
\end{array}$$

The three columns are biequivalences (the first one, between internal groupoids and internal crossed modules, is due to G. Janelidze, see [15]) and require $\mathcal{C}$ to be semi-abelian; the homomorphisms called $\mathcal{F}$ are bicategories of fractions, the lower one requires $\mathcal{C}$ to be semi-abelian and the upper one holds when $\mathcal{C}$ is just efficiently regular. As an intermediate step in order to establish the universal property of $\mathbf{Fract}(\mathcal{C})$, in Proposition 5.7, we characterize various kinds of representable profunctors in terms of fractors.

2. Preliminaries on profunctors

Profunctors, introduced by J. Bénabou in [4] under the name of *distributeurs* (see [5] for a more recent account), are a fruitful generalization to categories of the notion of relation, they provide an interesting formal approach to category theory (see [7]) and, together with functors, constitute a fundamental example of pseudo double category (see [14].

A profunctor $\mathbb{H} \nrightarrow \mathbb{G}$ is a functor $\mathbb{H} \times \mathbb{G}^{op} \to \mathbf{Set}$. Equivalently, a profunctor can be described in terms of a discrete fibration over $\mathbb{H}$ and a discrete cofibration over $\mathbb{G}$. The latter definition is internal (see [18]) and we recall it here in details. We assume the reader familiar with internal categories (and groupoids), internal functors and internal natural transformations in a finitely complete category $\mathcal{C}$ (see [8], Chapter 8, for an introduction).

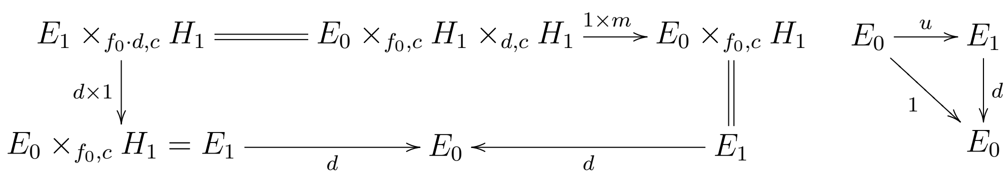
2.1 An internal functor $f = (f_0, f_1)\colon \mathbb{E} \to \mathbb{H}$ in $\mathcal{C}$

$$\begin{array}{ccc}
E_1 & \overset{c}{\underset{d}{\rightrightarrows}} \ \overset{u}{\leftarrow} & E_0 \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\
H_1 & \overset{c}{\underset{d}{\rightrightarrows}} \ \overset{u}{\leftarrow} & H_0
\end{array}$$

is a *discrete fibration* if the square

$$c \cdot f_1 = f_0 \cdot c$$

is a pullback (which implies that also the square $f_1 \cdot u = u \cdot f_0$ is a pullback). When this is the case, the domain map $d\colon E_1 \to E_0$ is a *right action* of $\mathbb{H}$ on E_0, that is, the diagrams



commute (where m is the composition in $\mathbb{H}$).
Conversely, given an internal category $\mathbb{H}$ and a split pullback

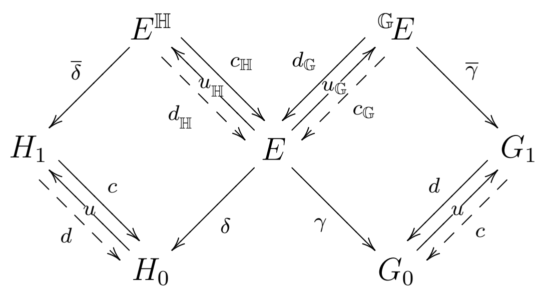
$$\begin{array}{ccc} E_1 & \underset{u}{\overset{c}{\rightleftarrows}} & E_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ H_1 & \underset{u}{\overset{c}{\rightleftarrows}} & H_0 \end{array}$$

if there exists an arrow $d\colon E_1 \to E_0$ making commutative the two previous diagrams, then $f = (f_0, f_1)\colon \mathbb{E} \to \mathbb{H}$ is a discrete fibration between internal categories, the composition in $\mathbb{E}$ being defined in the obvious way via the universal property of E_1.
The same relation holds between a *discrete cofibration*, that is, an internal functor $f\colon \mathbb{E} \to \mathbb{H}$ for which the square

$$d \cdot f_1 = f_0 \cdot d$$

is a pullback, and a *left action* of $\mathbb{H}$ on E_0.

Definition 2.2 Let $\mathcal{C}$ be a finitely complete category. A *profunctor* $E\colon \mathbb{H} \nrightarrow \mathbb{G}$ is given by a discrete fibration on $\mathbb{H}$ and a discrete cofibration on $\mathbb{G}$ as in the diagram



(1)

These data must satisfy two conditions:

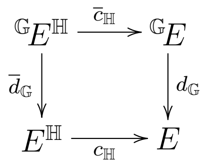
C1 The two actions on E are *compatible*:

$$\gamma \cdot d_{\mathbb{H}} = \gamma \cdot c_{\mathbb{H}} \qquad \delta \cdot d_{\mathbb{G}} = \delta \cdot c_{\mathbb{G}}$$

C2 The two actions on E *commute*:

$$d_{\mathbb{H}} \cdot n_{\mathbb{H}} = c_{\mathbb{G}} \cdot n_{\mathbb{G}}$$

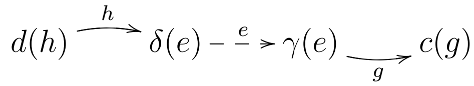
where the arrows $n_{\mathbb{H}}$ and $n_{\mathbb{G}}$ are defined as follows: consider the pullback



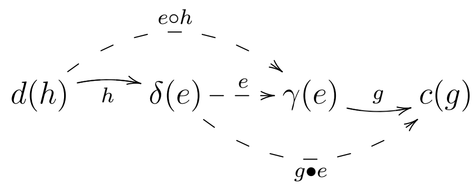
when **C1** holds, one can define

- $n_{\mathbb{H}} \colon {}^{\mathbb{G}}E^{\mathbb{H}} \to E^{\mathbb{H}}$ as the unique arrow such that $\overline{\delta} \cdot n_{\mathbb{H}} = \overline{\delta} \cdot \overline{d}_{\mathbb{G}}$ and $c_{\mathbb{H}} \cdot n_{\mathbb{H}} = c_{\mathbb{G}} \cdot \overline{c}_{\mathbb{H}}$
- $n_{\mathbb{G}} \colon {}^{\mathbb{G}}E^{\mathbb{H}} \to {}^{\mathbb{G}}E$ as the unique arrow such that $\overline{\gamma} \cdot n_{\mathbb{G}} = \overline{\gamma} \cdot \overline{c}_{\mathbb{H}}$ and $d_{\mathbb{G}} \cdot n_{\mathbb{G}} = d_{\mathbb{H}} \cdot \overline{d}_{\mathbb{G}}$

2.3 In the definition of profunctor, if we look at the elements of E as *virtual arrows*, we can consider a profunctor as a kind of generalized category, where it is possible to compose the virtual arrow e on the right with (composable) arrows of $\mathbb{H}$ and on the left with (composable) arrows of $\mathbb{G}$:

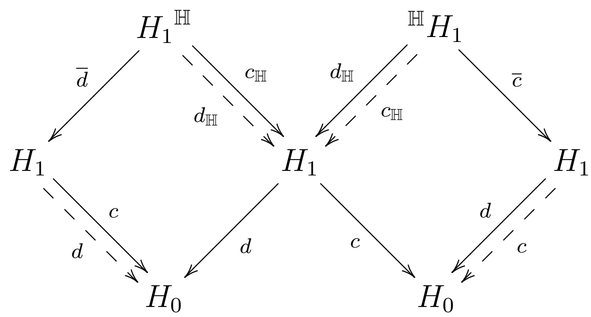


Condition **C1** says that the virtual codomain of the composition $e \circ h$ is exactly $\gamma(e)$ and the virtual domain of the composition $g \bullet e$ is $\delta(e)$, so that we can compare $g \bullet (e \circ h)$ with $(g \bullet e) \circ h$:



According to this set-theoretical description, condition **C2** simply states that the two compositions are equal, so that one can consider **C2** as a sort of mixed associativity axiom. The following example further explains this point of view.

Example 2.4 Any internal category $\mathbb{H}$ can be seen as a profunctor, by means of Yoneda:



where the two actions involved are the compositions on the right and on the left in $\mathbb{H}$. In this case, condition **C1** defines the partial multiplicative structure of the given category, while **C2** amounts to the associativity of the composition.

2.5 Given two profunctors with the same domain and codomain, it is natural to define a notion of morphism between them.
For $E\colon \mathbb{H} \nrightarrow \mathbb{G}$ and $E'\colon \mathbb{H} \nrightarrow \mathbb{G}$ with

$$H_0 \xleftarrow{\ \delta\ } E \xrightarrow{\ \gamma\ } G_0 \quad \text{and} \quad H_0 \xleftarrow{\ \delta'\ } E' \xrightarrow{\ \gamma'\ } G_0$$

respectively, a morphism $t\colon E \to E'$ is just a map t in $\mathcal{C}$ commuting with the δ's and the γ's

$$\delta' \cdot t = \delta \qquad \gamma' \cdot t = \gamma$$

and with the actions

$$t \cdot d_{\mathbb{H}} = d'_{\mathbb{H}} \cdot \langle \overline{\delta}, t \cdot c_{\mathbb{H}} \rangle \qquad t \cdot c_{\mathbb{G}} = c'_{\mathbb{G}} \cdot \langle t \cdot d_{\mathbb{G}}, \overline{\gamma} \rangle$$

It is not hard to show that this defines a (hom-)category $\mathbf{Prof}(\mathcal{C})(\mathbb{H}, \mathbb{G})$.

Remark 2.6 It is possible to associate to any profunctor $E\colon \mathbb{H} \nrightarrow \mathbb{G}$ a span

$$\mathbb{H} \xleftarrow{\ w\ } \mathbb{E} \xrightarrow{\ v\ } \mathbb{G}$$

of categories and functors in the following way (see Section 1 in [11] for more details): consider the pullback

$$\begin{array}{ccc} E_1 & \xrightarrow{\ \pi_{\mathbb{H}}\ } & {}^{\mathbb{G}}E \\ {\scriptstyle \pi_{\mathbb{G}}}\downarrow & & \downarrow{\scriptstyle c_{\mathbb{G}}} \\ E^{\mathbb{H}} & \xrightarrow[\ d_{\mathbb{H}}\ ]{} & E \end{array}$$

the span is given by

$$\begin{array}{ccccc} H_1 & \xleftarrow{\ \overline{\delta}\cdot\pi_{\mathbb{G}}\ } & E_1 & \xrightarrow{\ \overline{\gamma}\cdot\pi_{\mathbb{H}}\ } & G_1 \\ {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} & & {\scriptstyle d_{\mathbb{G}}\cdot\pi_{\mathbb{H}}}\downarrow\downarrow{\scriptstyle c_{\mathbb{H}}\cdot\pi_{\mathbb{G}}} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ H_0 & \xleftarrow[\ \delta\ ]{} & E & \xrightarrow[\ \gamma\ ]{} & G_0 \end{array}$$

Remark 2.7 Since in the next section we will deal with profunctors in a Maltcev category, let us recall that, if we restrict our attention to internal groupoids, then

1. discrete fibrations coincide with discrete cofibrations;

2. in Definition 2.2, the squares

$$c_{\mathbb{H}} \cdot \overline{d}_{\mathbb{G}} = d_{\mathbb{G}} \cdot \overline{c}_{\mathbb{H}}\ ,\ \ d_{\mathbb{H}} \cdot n_{\mathbb{H}} = c_{\mathbb{G}} \cdot n_{\mathbb{G}}\ ,\ \ d_{\mathbb{H}} \cdot \pi_{\mathbb{G}} = c_{\mathbb{G}} \cdot \pi_{\mathbb{H}}\ ,\ \ c_{\mathbb{H}} \cdot n_{\mathbb{H}} = c_{\mathbb{G}} \cdot \overline{c}_{\mathbb{H}}\ ,\ \ d_{\mathbb{G}} \cdot n_{\mathbb{G}} = d_{\mathbb{H}} \cdot \overline{d}_{\mathbb{G}}$$

are isomorphic pullbacks;

3. it is possible to flip a profunctor exchanging the role of domain and codomain. We denote this operation $()^{op}$:

$$E\colon \mathbb{H} \nrightarrow \mathbb{G} \quad \mapsto \quad E^{op}\colon \mathbb{G} \nrightarrow \mathbb{H}$$

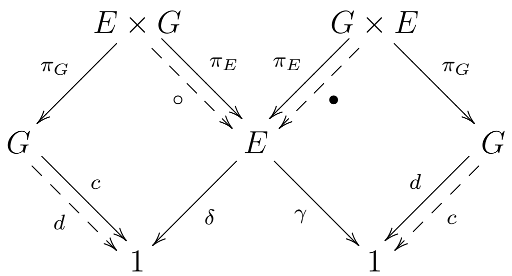
2.8 Finally, let us recall that, when the base category $\mathcal{C}$ is Barr-exact (see [11]), profunctors compose

$$E\colon \mathbb{H} \nrightarrow \mathbb{G}\ ,\ E'\colon \mathbb{G} \nrightarrow \mathbb{K} \quad \mapsto \quad E' \cdot E\colon \mathbb{H} \nrightarrow \mathbb{K}$$

(composition of profunctors is modelled on the tensor product of bimodules, see [4]) and it is not difficult to see that internal categories, internal profunctors and their morphisms form a bicategory $\mathbf{Prof}(\mathcal{C})$, the identity profunctor being the one described in Example 2.4. The interested reader may look at [18] for a full treatment or at [11] for a detailed account.

3. Profunctors in the Maltcev context

It is clear that condition **C1** of the definition of profunctors is necessary in order to merely state condition **C2**. It is natural to ask if it is also sufficient. The answer in general is negative. Actually two compatible actions usually need not to commute, so that they do not give rise to a profunctor.
An easy example is given by a right action $\circ\colon E \times G \to E$ of a group G on a set E. By taking as a left action $g \bullet e = e \circ g^{-1}$, we do have two compatible actions:



It is an easy exercise to show that these actions do not commute in general (they commute when the group G is abelian).

Now suppose that the category $\mathcal{C}$ is Maltcev, i.e., a finitely complete category in which any reflexive internal relation is an equivalence relation [13]. An important feature of Maltcev categories is that every internal category is a groupoid. Actually more is true. As shown by A. Carboni, M.C. Pedicchio and N. Pirovano in [13], in order to have an internal category, it is not necessary to impose the associativity axiom: this comes for free from

the multiplicative structure. Furthermore, if a reflexive graph admits a multiplicative structure, it is unique, so that internal functors are just morphisms of the underlying reflexive graphs.
Example 2.4 shows that, when an internal category $\mathbb{H}$ is seen as a profunctor, condition **C2** corresponds to the associativity of the composition, and associativity comes for free in the Maltcev context. This suggests the following result.

Proposition 3.1 *Let $\mathcal{C}$ be a Maltcev category. Condition* **C2** *in the definition of profunctor (Definition 2.2) follows from the other conditions.*

Proof. Consider the pullback involved in condition **C2**

$$\begin{array}{ccc} {}^{\mathbb{G}}E^{\mathbb{H}} & \xrightarrow{\overline{c}_{\mathbb{H}}} & {}^{\mathbb{G}}E \\ {\scriptstyle \overline{d}_{\mathbb{G}}}\downarrow & & \downarrow{\scriptstyle d_{\mathbb{G}}} \\ E^{\mathbb{H}} & \xrightarrow[c_{\mathbb{H}}]{} & E \end{array}$$

The sections $u_{\mathbb{H}}$ of $c_{\mathbb{H}}$ and $u_{\mathbb{G}}$ of $d_{\mathbb{G}}$ give rise to sections $\overline{u}_{\mathbb{H}}$ of $\overline{c}_{\mathbb{H}}$ and $\overline{u}_{\mathbb{G}}$ of $\overline{d}_{\mathbb{G}}$ such that

$$u_{\mathbb{G}} \cdot c_{\mathbb{H}} = \overline{c}_{\mathbb{H}} \cdot \overline{u}_{\mathbb{G}}\,;\,, \quad u_{\mathbb{H}} \cdot d_{\mathbb{G}} = \overline{d}_{\mathbb{G}} \cdot \overline{u}_{\mathbb{H}}\ , \quad \overline{u}_{\mathbb{G}} \cdot u_{\mathbb{H}} = \overline{u}_{\mathbb{H}} \cdot u_{\mathbb{G}}$$

Therefore, the diagram

$$\begin{array}{ccccc} E^{\mathbb{H}} & \xleftarrow{\overline{d}_{\mathbb{G}}} & {}^{\mathbb{G}}E^{\mathbb{H}} & \xrightarrow{\overline{c}_{\mathbb{H}}} & {}^{\mathbb{G}}E \\ {\scriptstyle c_{\mathbb{H}}}\downarrow\uparrow{\scriptstyle u_{\mathbb{H}}} & & {\scriptstyle c_{\mathbb{H}}\cdot\overline{d}_{\mathbb{G}}}\downarrow\uparrow{\scriptstyle \overline{u}_{\mathbb{G}}\cdot u_{\mathbb{H}}} & & {\scriptstyle d_{\mathbb{G}}}\downarrow\uparrow{\scriptstyle u_{\mathbb{G}}} \\ E & = & E & = & E \end{array}$$

is a product in the fibre $\mathcal{P}t_E(\mathcal{C})$ of the fibration of points. Since $\mathcal{C}$ is Maltcev, $\mathcal{P}t_E(\mathcal{C})$ is unital, so that the injections

$$E^{\mathbb{H}} \xrightarrow{\overline{u}_{\mathbb{G}}} {}^{\mathbb{G}}E^{\mathbb{H}} \xleftarrow{\overline{u}_{\mathbb{H}}} {}^{\mathbb{G}}E$$

are jointly (strongly) epimorphic (see [9], Chapter 2). Now we check condition **C2** precomposing with $\overline{u}_{\mathbb{G}}$ and $\overline{u}_{\mathbb{H}}$. We examine the calculation when precomposing with $\overline{u}_{\mathbb{H}}$, the other one being similar. We need three steps:
First step: $n_{\mathbb{G}} \cdot \overline{u}_{\mathbb{H}} = 1$. For this, just compose with the pullback projections

$$G_1 \xleftarrow{\overline{\gamma}} {}^{\mathbb{G}}E \xrightarrow{d_{\mathbb{G}}} E$$

Second step: $n_{\mathbb{H}} \cdot \overline{u}_{\mathbb{H}} = u_{\mathbb{H}} \cdot c_{\mathbb{G}}$. For this, compose with the pullback projections

$$H_1 \xleftarrow{\overline{\delta}} E^{\mathbb{H}} \xrightarrow{c_{\mathbb{H}}} E$$

and use condition **C1** when composing with $\overline{\delta}$.
Third step: using the first and the second step, we have

$$c_{\mathbb{G}} \cdot n_{\mathbb{G}} \cdot \overline{u}_{\mathbb{H}} = c_{\mathbb{G}} = d_{\mathbb{H}} \cdot u_{\mathbb{H}} \cdot c_{\mathbb{G}} = d_{\mathbb{H}} \cdot n_{\mathbb{H}} \cdot \overline{u}_{\mathbb{H}}$$

■

4. Crossed profunctors

A semi-abelian category is a Barr-exact, pointed and protomodular category with binary coproducts, see [16] or [9]. Semi-abelian categories satisfy the Maltcev condition. Moreover, we assume that the condition "Huq is Smith" holds, see [21], and call such a category H-S semi-abelian.

4.1 In a H-S semi-abelian category, an internal crossed module $\mathbb{H}$ is given by two arrows

$$H_0\flat H \xrightarrow{\xi_{\mathbb{H}}} H \xrightarrow{\partial_{\mathbb{H}}} H_0$$

making commutative the diagram

$$\begin{array}{ccc} H\flat H & \xrightarrow{\chi_H} & H \\ {\scriptstyle \partial_{\mathbb{H}}\flat 1}\downarrow & & \downarrow{\scriptstyle 1} \\ H_0\flat H & \xrightarrow{\xi_{\mathbb{H}}} & H \\ {\scriptstyle 1\flat\partial_{\mathbb{H}}}\downarrow & & \downarrow{\scriptstyle \partial_{\mathbb{H}}} \\ H_0\flat H_0 & \xrightarrow[\chi_{H_0}]{} & H_0 \end{array}$$

where χ_X is the canonical conjugation action on an object X, and $X\flat Y$ is the object part of the kernel of $[1,0]\colon X+Y\to X$ (see [20] for a detailed account). A morphism $\mathbb{H}\to\mathbb{G}$ of crossed modules is a pair of arrows $H\to G$ and $H_0\to G_0$ commuting with the ∂'s and the ξ's.
In [15], G. Janelidze proved that the category $\mathbf{Grpd}(\mathcal{C})$ of internal groupoids and internal functors is equivalent to the category $\mathbf{XMod}(\mathcal{C})$ of internal crossed modules (in fact, this is a biequivalence of bicategories, see Corollary 2.12 in [1]). The process of associating a crossed module to a groupoid is called *normalization*: given a groupoid $\mathbb{H}$

$$H_1 \overset{c}{\underset{d}{\underset{\xleftarrow{u}}{\rightrightarrows}}} H_0$$

we get an action $\xi\colon H_0\flat H\to H$ by the following diagram, where the rows are kernels,

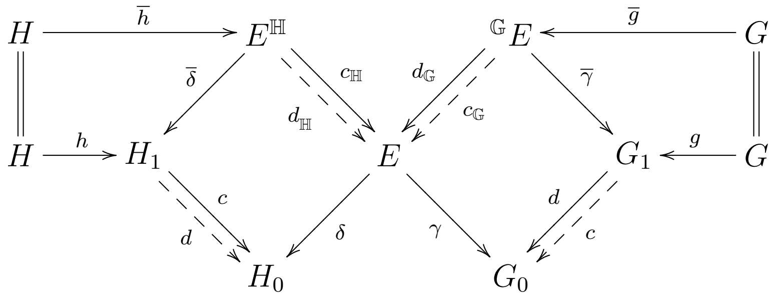
$$\begin{array}{ccccc} H_0\flat H & \longrightarrow & H_0+H & \xrightarrow{[1,0]} & H_0 \\ {\scriptstyle \xi_{\mathbb{H}}}\downarrow & & \downarrow{\scriptstyle [u,h]} & & \downarrow{\scriptstyle 1} \\ H & \xrightarrow[h]{} & H_1 & \xrightarrow[d]{} & H_0 \end{array}$$

The crossed module associated to $\mathbb{H}$ is then

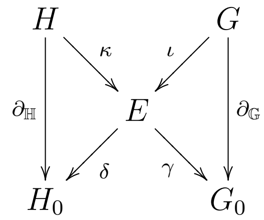
$$H_0\flat H \xrightarrow{\xi_{\mathbb{H}}} H \xrightarrow{\partial_{\mathbb{H}}=c\cdot h} H_0$$

A notational convention: we will often use the same notation for groupoids and their associated crossed modules, so that $\mathbb{H}$ stays for $(\partial_{\mathbb{H}},\xi_{\mathbb{H}})$ and for the associated groupoid (H_1,H_0).

4.2 We apply now the normalization process to the four groupoids involved in the definition of profunctor: using the notation of Definition 2.2, we first get



(recall that d and $d_{\mathbb{H}}$ have same kernel because $d_{\mathbb{H}}$ is the pullback of d along δ, and the same holds for d and $d_{\mathbb{G}}$), and then we get a commutative diagram



(2)

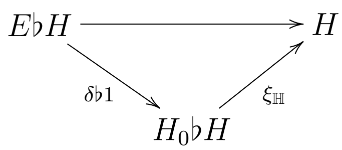
where $\kappa = c_{\mathbb{H}} \cdot \overline{h}$ and $\iota = c_{\mathbb{G}} \cdot \overline{g}$.

Lemma 4.3 *Let $\mathcal{C}$ be a H-S semi-abelian category. A profunctor $E \colon \mathbb{H} \looparrowright G$ yields the commutative diagram (2), where $\partial_{\mathbb{H}}$ and $\partial_{\mathbb{G}}$ are crossed modules and the following conditions are fulfilled:*

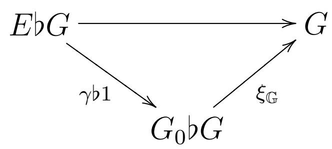
i. $\gamma \cdot \kappa = 0$,

ii. $\delta \cdot \iota = 0$

iii. The action of the crossed module $\kappa \colon H \to E$ is compatible with the action of the crossed module $\partial_{\mathbb{H}} \colon H \to H_0$, that is, the following diagram commutes



iv. The action of the crossed module $\iota \colon G \to E$ is compatible with the action of the crossed module $\partial_{\mathbb{G}} \colon G \to G_0$, that is, the following diagram commutes



Conversely, given such a diagram satisfying conditions (i) to (iv) above, one recovers a profunctor $\mathbb{H} \looparrowright \mathbb{G}$.

Proof. The first sentence is proved just by simple computations. Conversely, it is easy to see that the commutativity of the left triangle in diagram (2), plus condition (*iii*), makes the pair $(1_H, \delta)$

$$\begin{array}{ccc} H & \xrightarrow{1_H} & H \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\partial_{\mathbb{H}}} \\ E & \xrightarrow[\delta]{} & H_0 \end{array}$$

a discrete fibration of crossed modules, and therefore a discrete fibration of the associated groupoids (see [1], Remark 3.2), and similarly for the right triangle and condition (*iv*). Moreover, conditions (*i*) and (*ii*) imply condition **C1** of Definition 2.2 on the associated groupoids, thanks to the protomodularity of $\mathcal{C}$. Finally, since $\mathcal{C}$ is Maltcev, condition **C2** of Definition 2.2 follows from Proposition 3.1. ∎

The previous result justifies the following definition, which extends to the semi-abelian context a notion introduced by M. Jibladze in [17] in the case of groups.

Definition 4.4 Let $\mathcal{C}$ be a H-S semi-abelian category, and consider two internal crossed modules $\mathbb{H}$ and $\mathbb{G}$. A *crossed profunctor* $E\colon \mathbb{H} \looparrowright \mathbb{G}$ is a commutative diagram of the form

$$\begin{array}{ccccc} H & & & & G \\ & \searrow^{\kappa} & & {}^{\iota}\swarrow & \\ {\scriptstyle\partial_{\mathbb{H}}}\downarrow & & E & & \downarrow{\scriptstyle\partial_{\mathbb{G}}} \\ & \swarrow_{\delta} & & {}_{\gamma}\searrow & \\ H_0 & & & & G_0 \end{array}$$

such that

i. $\gamma \cdot \kappa = 0$,

ii. $\delta \cdot \iota = 0$

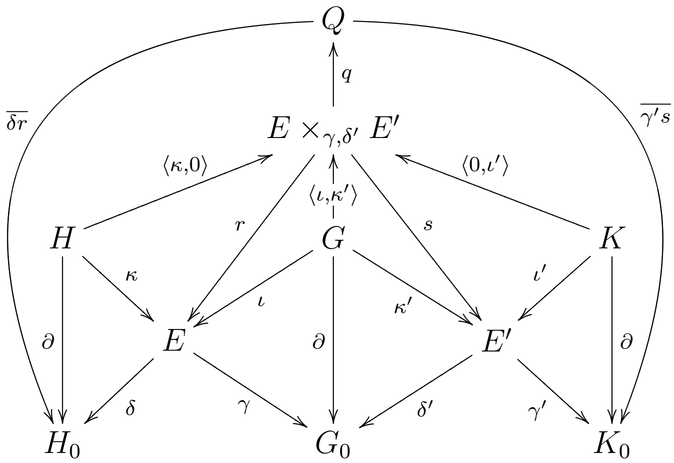
iii. The action of E on H induced by that of H_0 on H via δ makes $\kappa\colon H \to E$ a (pre)crossed module

iv. The action of E on G induced by that of G_0 on G via γ makes $\iota\colon g \to E$ a (pre)crossed module.

A morphism of crossed profunctors $E, E'\colon \mathbb{H} \looparrowright \mathbb{G}$ is an arrow $f\colon E \to E'$ commuting with the κ's, the ι's, the δ's and the γ's.

4.5 In order to obtain a bicategory $\mathbf{XProf}(\mathcal{C})$ of crossed modules and crossed profunctors, we describe now the composition of crossed profunctors.

Let us consider two crossed profunctors $E\colon \mathbb{H} \nrightarrow \mathbb{G}$ and $E'\colon \mathbb{G} \nrightarrow \mathbb{K}$. The composite $E' \cdot E\colon \mathbb{H} \nrightarrow \mathbb{K}$ is defined by the following construction:



The central object Q of the composite $E' \cdot E$ is obtained by first pulling-back (E, γ) and (E', δ'), then taking the quotient over the normal subobject $(G, \langle \iota, \kappa' \rangle)$. The four morphisms that give the crossed profunctor $E' \cdot E$ are $q \cdot \langle \kappa, 0 \rangle$, $q \cdot \langle 0, \iota' \rangle$, and $\overline{\delta r}$, $\overline{\gamma' s}$. The first two are obtained by the universal property of the pullback $E \times_{\gamma,\delta'} E'$, the last two by the universal property of the cokernel Q. Calculations show that, even if not associative on the nose, this composition is coherently weakly associative.
For each crossed module $\mathbb{H}$, its identity crossed profunctor is precisely the normalization of the identity profunctor of the groupoid corresponding to $\mathbb{H}$ (see Example 2.4).
It is not difficult to show that the composition just defined extends functorially to 2-cells, and that these data form a bicategory $\mathbf{XProf}(\mathcal{C})$.

Now we can complete the comparison between profunctors and crossed profunctors, started in Lemma 4.3, making precise the idea that crossed profunctors are the normalized version of profunctors.

Proposition 4.6 *Let $\mathcal{C}$ be a H-S semi-abelian category. The normalization process extends to a biequivalence of bicategories*

$$\mathbf{Prof}(\mathcal{C}) \simeq \mathbf{XProf}(\mathcal{C})$$

Proof. The normalization process clearly determines a homomorphism of bicategories. In fact it is straightforward (even if cumbersome) to show that the composition of the normalization is (isomorphic to) the normalization of the composition. Moreover, the similar property concerning the identities follows from the very definition. Coherence is granted by the universal properties involved. This homomorphism clearly yields equivalences on the hom-categories

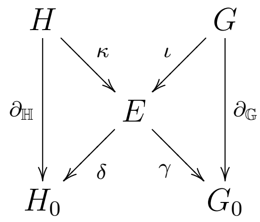
$$\mathbf{Prof}(\mathcal{C})(\mathbb{H}, \mathbb{G}) \to \mathbf{XProf}(\mathcal{C})(\mathbb{H}, \mathbb{G}).$$

Finally, the homomorphism is not just biessentially surjective, but essentially surjective. This is a consequence of the equivalence between the category of groupoids and that of crossed modules. ∎

5. Butterflies and fractors

Among crossed profunctors, of special interest are butterflies, introduced in [22] in the case of groups and extended to the semi-abelian context in [1].

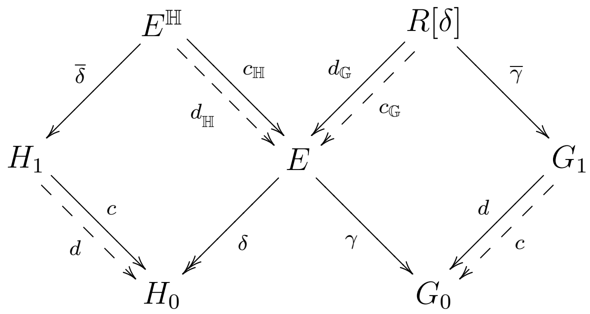
Definition 5.1 Let $\mathbb{H}$ and $\mathbb{G}$ be crossed modules in a H-S semi-abelian category $\mathcal{C}$. A crossed profunctor $E\colon \mathbb{H} \nrightarrow \mathbb{G}$



is a *butterfly* if the NE-SW complex is an extension: δ is the cokernel of ι and ι is the kernel of δ.

Butterflies form a locally groupoidal sub-bicategory $\mathbf{Bfly}(\mathcal{C})$ of $\mathbf{XProf}(\mathcal{C})$ (see Section 3.7 in [1] for the proof that butterflies are closed in $\mathbf{XProf}(\mathcal{C})$ under composition).

A natural question arises. What happens if we "denormalize" butterflies, i.e. we go back to groupoids via the equivalence between $\mathbf{Grpd}(\mathcal{C})$ and $\mathbf{XMod}(\mathcal{C})$?
We obtain the following diagram:



(3)

where the NE-SW fork is an exact fork. This yields a new notion. More precisely:

Definition 5.2 A *fractor* in a category $\mathcal{C}$ with finite limits is a pair of left-right compatible actions of groupoids over an object E (as in diagram (3) above) where the NE-SW fork is an exact fork, i.e. δ is a regular epimorphism and $(d_{\mathbb{G}}, c_{\mathbb{G}})$ is its kernel pair.

Observe that it is not required to be a profunctor. Actually it is so: the commutativity of the two actions comes for free even if the base category is *not* Maltcev (see [1], Remark 3.5). The name is justified by the fact that fractors form the bicategory of fractions of the 2-category of internal groupoids with respect to weak equivalences (see Theorem 6.2).

Remark 5.3 In a finitely complete $\mathcal{C}$, fractors give rise to those profunctors between groupoids independently considered by D. Bourn in [11] as the ones whose canonical span representation has a fully faithful, surjective on objects, left leg. In fact we will identify fractors with these, when no confusion arises.

When $\mathcal{C}$ is efficiently regular, fractors form a locally groupoidal bicategory $\mathbf{Fract}(\mathcal{C})$. This forms a sub-bicategory of $\mathbf{Prof}(\mathcal{C})$ when profunctor composition can be defined, i.e. for instance when $\mathcal{C}$ is Barr-exact (see [11]).

Proposition 5.4 *Let $\mathcal{C}$ be a H-S semi-abelian category. The biequivalence*

$$\mathbf{Prof}(\mathcal{C}) \simeq \mathbf{XProf}(\mathcal{C})$$

of Proposition 4.6 restricts to a biequivalence

$$\mathbf{Fract}(\mathcal{C}) \simeq \mathbf{Bfly}(\mathcal{C})$$

Proof. See Section 3.3 in [1]. ■

From Proposition 5.4 and [1], Theorem 5.8, we have that, in the semi-abelian context, $\mathbf{Fract}(\mathcal{C})$ is the bicategory of fractions of $\mathbf{Grpd}(\mathcal{C})$ with respect to weak equivalences. In order to prove that this result holds in the more general context of efficiently regular categories, we characterize now various kinds of representable profunctors in terms of fractors.

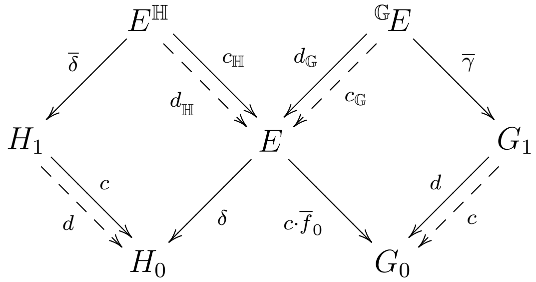
5.5 Recall from [18] that any internal functor $f\colon \mathbb{H} \to \mathbb{G}$ between internal categories gives rise to a profunctor $f_\bullet\colon \mathbb{H} \nrightarrow \mathbb{G}$ as follows: given an internal functor

$$\begin{array}{ccc} H_1 & \underset{d}{\overset{c}{\rightrightarrows}} & H_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ G_1 & \underset{d}{\overset{c}{\rightrightarrows}} & G_0 \end{array}$$

consider the pullback

$$\begin{array}{ccc} E & \xrightarrow{\overline{f}_0} & G_1 \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle d} \\ H_0 & \xrightarrow[f_0]{} & G_0 \end{array}$$

and define the profunctor $f_\bullet$ by



Moreover $f_\bullet$ has a right adjoint $f^\bullet \colon \mathbb{G} \nrightarrow \mathbb{H}$ in the bicategory $\mathbf{Prof}(\mathcal{C})$. If $\mathbb{G}$ and $\mathbb{H}$ are groupoids, $f^\bullet$ is nothing but $(f_\bullet)^{op}$. Recall also that, if $E\colon \mathbb{H} \nrightarrow \mathbb{G}$ is a profunctor and

$$\mathbb{H} \xleftarrow{\ w\ } \mathbb{E} \xrightarrow{\ v\ } \mathbb{G}$$

is its associated span as in 2.6, then $E \simeq v_\bullet \cdot w^\bullet$. Moreover, if $E\colon \mathbb{H} \nrightarrow \mathbb{G}$ is a fractor, it is proved in [11] that $w\colon \mathbb{E} \to \mathbb{H}$ is a weak equivalence in the sense of [12], that is, a functor which is internally fully faithful and essentially surjective on objects.

A profunctor $E\colon \mathbb{H} \nrightarrow \mathbb{G}$ is *representable* if it is isomorphic to $f_\bullet$ for some internal functor $f\colon \mathbb{H} \to \mathbb{G}$.

Lemma 5.6 *Let $\mathcal{C}$ be a Barr-exact category and $E\colon \mathbb{H} \nrightarrow \mathbb{G}$ a profunctor between internal groupoids. If E is a fractor, then it has a right adjoint given by E^{op}.*

Proof. Starting from a fractor $E\colon \mathbb{H} \nrightarrow \mathbb{G}$, we know that in its canonical representation as a span of functors

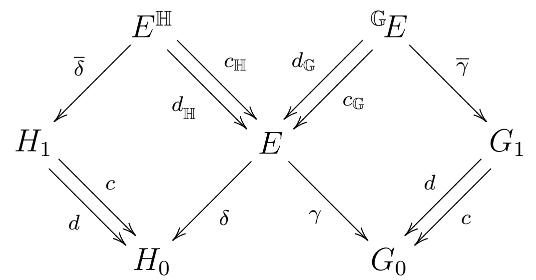
$$\mathbb{H} \xleftarrow{\ w\ } \mathbb{E} \xrightarrow{\ v\ } \mathbb{G}$$

the left leg w is a weak equivalence. Moreover, as for any profunctor, $E \simeq v_\bullet \cdot w^\bullet$. In this setting, one can easily recover the counit and the unit of the adjunction $E \dashv E^{op}$

$$\begin{aligned} \epsilon &: E \cdot E^{op} \simeq v_\bullet \cdot w^\bullet \cdot (v_\bullet \cdot w^\bullet)^{op} \simeq v_\bullet \cdot w^\bullet \cdot (w^\bullet)^{op} \cdot (v_\bullet)^{op} \simeq v_\bullet \cdot w^\bullet \cdot w_\bullet \cdot v^\bullet \simeq v_\bullet \cdot v^\bullet \Rightarrow 1 \\ \eta &: 1 \simeq w_\bullet \cdot w^\bullet \Rightarrow w_\bullet \cdot v^\bullet \cdot v_\bullet \cdot w^\bullet \simeq E^{op} \cdot E \end{aligned}$$

from those of the adjunction $v_\bullet \dashv v^\bullet$ and of the adjoint equivalence $(w_\bullet, w^\bullet)$. ■

Proposition 5.7 *Let $\mathcal{C}$ be finitely complete and $E : \mathbb{H} \nrightarrow \mathbb{G}$ a profunctor between internal groupoids:*



(i) *E is representable if and only if it is a* split fractor*, that is a fractor with its left leg δ a split epimorphism.*

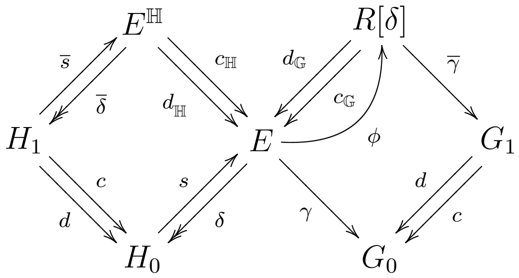
(ii) *E is representable by an essentially surjective functor if and only if it is a split fractor with its right leg γ a regular epimorphism.*

(iii) *E is representable by a fully faithful functor if and only if it is a split fractor with $(E^{\mathbb{H}}, d_{\mathbb{H}}, c_{\mathbb{H}})$ the kernel pair of γ.*

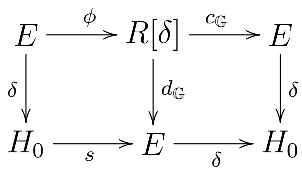
(iv) E is representable by a weak equivalence if and only if it is a split fractor with E^{op} still a fractor.

(v) In the case $\mathcal{C}$ is Barr-exact, a profunctor $E\colon \mathbb{H} \nrightarrow \mathbb{G}$ is representable by an equivalence if and only if it is a split fractor with E^{op} still a split fractor.

Proof. (i): Given a functor f, the left leg δ of $f_\bullet$ clearly splits, as it is the pullback along f_0 of the split epimorphism $d\colon G_1 \to G_0$. Conversely, let us consider a split fractor E:



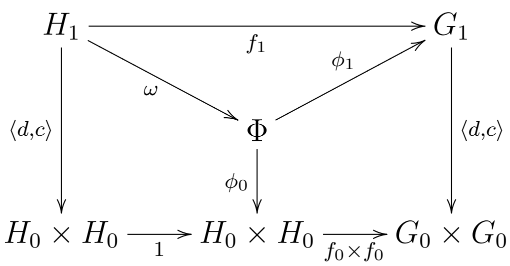
where the section $\overline{s}$ is the pullback of a section s, and ϕ is the only arrow such that $d_{\mathbb{G}} \cdot \phi = s \cdot \delta$ and $c_{\mathbb{G}} \cdot \phi = 1_E$. Moreover, in the diagram



the square on the left is a pullback, since the whole and the one on the right are.
One can easily prove that the assignments $f_0 = \gamma \cdot s$ and $f_1 = \overline{\gamma} \cdot \phi \cdot d_{\mathbb{H}} \cdot \overline{s}$ produce an internal functor $f = (f_0, f_1)$ with $f_\bullet = E$. ∎

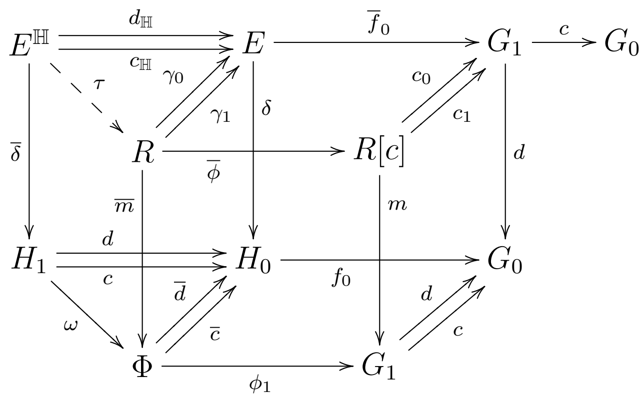
Proof. (ii): Since we are working in the context of (internal) groupoids, being essentially surjective for f (i.e. f_0 surjective up to isomorphisms) amounts to $c \cdot \overline{f_0}$ being a regular epimorphism (i.e. f_0 surjective up to morphisms!), but in the profunctorial representation $f_\bullet$ of f this last morphism is nothing but γ. ∎

Before we prove the other statements of Proposition 5.7, let us recall that any internal functor $f\colon \mathbb{H} \to \mathbb{G}$ can be factored into a bijective on objects functor followed by a fully faithful one. The corresponding construction is well-known (e.g. [11]):



The pullback (Φ, ϕ_0, ϕ_1) of $\langle d, c\rangle$ along $f_0 \times f_0$ gives rise to an internal category Φ with the same objects as $\mathbb{H}$. Then the pairs $(\omega, 1_{H_0})$ and (ϕ_1, f_0) yield the desired factorization. Observe that f itself is fully faithful precisely when w is an isomorphism, i.e., the outer rectangle above is a pullback.

Proof. (iii): Let us consider the following diagram, where E is the pullback of f_0 and d as in 5.5,



The functor $(\overline{\delta}, \delta)$ is the discrete fibration corresponding to the left leg of $f_\bullet$. Moreover the arrows ω and ϕ_1 are precisely those given by the factorization of f as described above. Since $\mathbb{G}$ is a groupoid, given a kernel pair $(R[c], c_0, c_1)$, the object $R[c]$ is also the pullback of c along d, with projections m and c_1. Now let us consider the pullback R of m along ϕ_1. We get the arrows $\gamma_0 = \langle c_0 \cdot \overline{\phi}, \overline{d} \cdot \overline{m}\rangle$ and $\gamma_1 = \langle c_1 \cdot \overline{\phi}, \overline{c} \cdot \overline{m}\rangle$ onto the pullback E. Moreover, by pullback composition, we have the morphism $\tau = \langle \omega \cdot \overline{\delta}, \overline{f} \cdot c_{\mathbb{H}}\rangle : E^{\mathbb{H}} \to R$ that satisfies $\gamma_0 \cdot \tau = d_{\mathbb{H}}$ and $\gamma_1 \cdot \tau = c_{\mathbb{H}}$.
Now it is not difficult to show that (R, γ_0, γ_1) is the kernel pair of $\gamma = c \cdot \overline{f}_0$. Finally f is fully faithful, i.e. ω is an isomorphism, if and only if τ is, whence the result. ■

Proof. (iv): Since an internal weak equivalence $f\colon \mathbb{H} \to \mathbb{G}$ is a functor which is fully faithful and essentially surjective, the result is achieved by (ii) and (iii). ■

Proof. (v): Let $E = f_\bullet$ with $f\colon \mathbb{H} \to \mathbb{G}$ an equivalence with quasi-inverse $g\colon \mathbb{G} \to \mathbb{H}$. Then g is (in particular) right adjoint to f and then $g_\bullet$ is right adjoint to $f_\bullet$. By Lemma 5.6 we also have $E \dashv E^{op}$ and then $E^{op} \simeq g_\bullet$. Since g is an equivalence, by point (iv) in Proposition 5.7 the profunctor E^{op} is a split fractor.
Conversely, assume that both E and E^{op} are split fractors. By point (iv) in Proposition 5.7, $E \simeq f_\bullet$ and $E^{op} \simeq g_\bullet$ with $f\colon \mathbb{H} \to \mathbb{G}$ and $g\colon \mathbb{G} \to \mathbb{H}$ two weak equivalences. Since $E \dashv E^{op}$ and $\mathcal{F}_{\mathbb{H},\mathbb{G}}$ is full and faithful, we get $f \dashv g$. Since f and g are fully faithful, the condition $f \dashv g$ immediately implies that f is an equivalence with quasi-inverse g. ■

The representation of functors inside fractors extends to an embedding

$$\mathcal{F}_{\mathbb{H},\mathbb{G}} \colon \mathbf{Grpd}(\mathcal{C})(\mathbb{H}, \mathbb{G}) \hookrightarrow \mathbf{Fract}(\mathcal{C})(\mathbb{H}, \mathbb{G}).$$

The discussion above describes this embedding on morphisms, while its definition on 2-cells is just the straightforward internalization of the following set-theoretical construction:

given a natural transformation $\alpha\colon\ (f_1, f_0) \Longrightarrow (g_1, g_0)$, the required isomorphism $E_f \to E_g$ is the map

$$\tilde{\alpha}\colon (h_0,\ f(h_0) \xrightarrow{g_1} g_0\ ) \ \mapsto\ (h_0,\ g(h_0) \xrightarrow{\alpha_{h_0}^{-1}} f(h_0) \xrightarrow{g_1} g_0\ )$$

6. Fractors are fractions

As a last preparatory step to prove that fractors are the bicategory of fractions, let us recall Proposition 2.5 from [11] (where a fractor E such that E^{op} is still a fractor is called a regularly fully faithful profunctor).

Lemma 6.1 *Let $\mathcal{C}$ be an efficiently regular category and $E\colon \mathbb{H} \nrightarrow \mathbb{G}$ a fractor between internal groupoids. If $E^{op}\colon \mathbb{G} \nrightarrow \mathbb{H}$ is still a fractor, then E is an equivalence in* $\mathbf{Fract}(\mathcal{C})$*, a quasi-inverse being necessarily given by E^{op}.*

In [23], D. Pronk has defined the bicategory of fractions of a bicategory $\mathcal{B}$ with respect to a class Σ of 1-cells. This is a homomorphism of bicategories

$$\mathcal{P}_\Sigma\colon \mathcal{B} \to \mathcal{B}[\Sigma^{-1}]$$

universal among all homomorphisms $\mathcal{F}\colon \mathcal{B} \to \mathcal{A}$ such that $\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$.

Theorem 6.2 *Let $\mathcal{C}$ be an efficiently regular category. The homomorphism of bicategories*

$$\mathcal{F}\colon \mathbf{Grpd}(\mathcal{C}) \to \mathbf{Fract}(\mathcal{C}) \quad (f\colon \mathbb{H} \to \mathbb{G}) \ \mapsto\ (f_\bullet\colon \mathbb{H} \nrightarrow \mathbb{G})$$

is the bicategory of fractions of $\mathbf{Grpd}(\mathcal{C})$ *with respect to the class Σ of weak equivalences.*

Proof. Since Σ admits a right calculus of fractions (see [25], Proposition 4.5), we can use Proposition 24 from [23] to prove that $\mathcal{F}$ is the bicategory of fractions.
- $\mathcal{F}(S)$ is an equivalence if S is a weak equivalence: this follows from point (iv) in Proposition 5.7 and Lemma 6.1.
- $\mathcal{F}$ is surjective on objects up to equivalence: obvious because $\mathcal{F}$ is the identity map on objects.
- $\mathcal{F}$ is full and faithful on 2-cells: we already noticed that $\mathcal{F}_{\mathbb{H},\mathbb{G}}$ is a full and faithful functor.
- For every fractor $E\colon \mathbb{H} \nrightarrow \mathbb{G}$ there exist internal functors $w\colon \mathbb{E} \to \mathbb{H}$ and $v\colon \mathbb{E} \to \mathbb{G}$ such that $w \in \Sigma$ and $E \cdot \mathcal{F}(w) \simeq \mathcal{F}(v)$: for this, consider once again the span of internal functors associated to the profunctor E

$$\mathbb{H} \xleftarrow{w} \mathbb{E} \xrightarrow{v} \mathbb{G}$$

We know that $w \in \Sigma$ (because E is a fractor) and that $E \simeq v_\bullet \cdot w^\bullet$ (see 5.5). Therefore:

$$E \cdot w_\bullet \simeq v_\bullet \cdot w^\bullet \cdot w_\bullet \simeq v_\bullet$$

∎

Remark 6.3 The year 2011 witnessed a debate on profunctors and anafunctors.
The notion of anafunctor has been introduced by M. Makkai in [19] in order to describe some *weak morphisms* of categories. Its internalization is due to T. Bartels [3]. Our interest in anafunctors is related to the fact that they constitute a bicategorical localization of internal categories with respect to internal weak equivalences, in the sense of [23], as proved by D. Roberts in [24].
We can compare the bicategory $\mathbf{Ana}(\mathcal{C})$ of anafunctors (with respect to the regular-epi Grothendieck pretopology) with $\mathbf{Fract}(\mathcal{C})$, for $\mathcal{C}$ a Barr-exact Maltcev category. By the universal property of the bicategory of fractions of the 2-category $\mathbf{Cat}(\mathcal{C})$, with respect to the class Σ of internal weak equivalences, one has the following chain of biequivalences

$$\mathbf{Fract}(\mathcal{C}) \simeq \mathbf{Cat}(C)[\Sigma^{-1}] \simeq \mathbf{Ana}(\mathcal{C}).$$

This establishes an internal version of J. Benabou's statement [6], namely that anafunctors precisely correspond to *locally representable profunctors*, i.e. in our terminology, *fractors*. This result is still valid when we drop the Maltcev hypothesis, provided we restrict our concern to internal groupoids instead of internal categories.

To end, let us observe that if in $\mathcal{C}$ the axiom of choice holds (that is, any regular epimorphism splits) then weak equivalences in $\mathbf{Grpd}(\mathcal{C})$ coincide with equivalences, and fractors coincide with representable profunctors (cf. Proposition 5.7), which are therefore the bicategory of fractions of $\mathbf{Grpd}(\mathcal{C})$ with respect to weak equivalences.

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