

Fibred and internal aspects of distributors composition

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- INTRODUCTION
- FIBRED ASPECTS OF DISTRIBUTORS COMPOSITION
- THE INTERNAL CASE

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Relations in categories

Let $\mathcal{C}$ be a finitely complete category, endowed with a $(\mathcal{E}, \mathcal{M})$ -orthogonal factorization system. A *relation* between objects A and B

$$A \xrightarrow[\perp]{S} B$$

is an $\mathcal{M}$ -subobject S of $A \times B$.

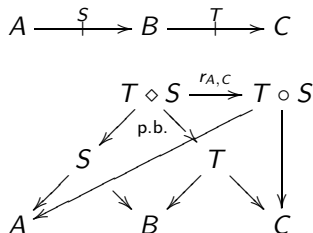
Relations between A and B give the category $\mathbf{Rel}(A, B)$

Every relation $S \subset A \times B$ determines a span: $A \longleftarrow S \longrightarrow B$, and the local embedding of relations into spans admits a reflection given by the $(\mathcal{E}, \mathcal{M})$ -factorization:

$$\mathbf{Rel}(A, B) \begin{array}{c} \xleftarrow{r_{A,B}} \\ \xrightarrow[\perp]{} \\ \xrightarrow{i_{A,B}} \end{array} \mathbf{Span}(A, B)$$

Relations in categories

Relations compose:



but the composition of relations is not associative, in general.

When the composition of relations in $\mathcal{C}$ is associative, relations form the bicategory

$$\mathbf{Rel}(\mathcal{C}) = \mathbf{Rel}(\mathcal{C}, (\mathcal{E}, \mathcal{M}))$$

of relations relative to the $(\mathcal{E}, \mathcal{M})$ -factorization system.

Example: For $\mathcal{C} = \mathbf{Set}$, $\mathcal{E} =$ surjections and $\mathcal{M} =$ injections,

Relations in categories

Theorem (Klein 1970, Kelly 1990)

If $\mathcal{M} \subset \text{Mono}$, then the composition of relation is associative iff the factorization system $(\mathcal{E}, \mathcal{M})$ is stable.

However, there are relevant examples of bicategories of relations relative to factorization system where the hypothesis $\mathcal{M} \subset \text{Mono}$ does not hold, and the class $\mathcal{E}$ is not stable under pullbacks.

Theorem (M. 2016, Distributors = Relators)

$$\mathbf{DistGpd} = \mathbf{Rel}(\mathbf{Gpd}, (\mathcal{F}, \mathcal{D}))$$

where $\mathcal{F}$ = final functors, and $\mathcal{D}$ = discrete fibrations.

Plan of the talk

1. **Internal groupoids:** if $\mathcal{E}$ is Barr-exact,

$$\mathbf{DistGpd}(\mathcal{E}) = \mathbf{Rel}(\mathbf{Gpd}(\mathcal{E}), (\mathcal{F}, \mathcal{D}))$$

2. **Categories:** distributor composition from a fibrational point of view.

... but we shall start from number 2.

Overview

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Definition

A two-sided fibration from $\mathbb{A}$ to $\mathbb{B}$ is a span of categories and functors

$$\mathbb{A} \xleftarrow{Q} \mathbb{E} \xrightarrow{P} \mathbb{B}$$

such that:

- (i) each arrow $b \xrightarrow{\beta} P(e')$ in $\mathbb{B}$ has a P -cartesian Q -vertical lift at e' ,
- (ii) each arrow $Q(e) \xrightarrow{\alpha} a$ in $\mathbb{A}$ has a Q -opcartesian P -vertical lift at e ,
- (iii) for every cartesian opcartesian composite $\beta^*e \longrightarrow e \longrightarrow \alpha_*e$, the canonical comparison $\alpha_*\beta^*e \longrightarrow \beta^*\alpha_*e$ is an isomorphism.

It is called *discrete* when it has discrete fibers (over $\mathbb{A} \times \mathbb{B}$).

$\mathbf{Fib}(\mathbb{A}, \mathbb{B})$ is the category of two-sided fibrations from $\mathbb{A}$ to $\mathbb{B}$.

$$\mathbf{opFib}(\mathbb{A}) = \mathbf{Fib}(\mathbb{A}, 1) \quad \mathbf{Fib}(\mathbb{B}) = \mathbf{Fib}(1, \mathbb{B})$$

Two factorization systems in **Cat**

The comprehensive factorization of a functor [Street and Walters 1973].

$$(\mathcal{D}, \mathcal{F}) \qquad (\mathcal{D}^{op}, \mathcal{F}^{op})$$

$$(\text{final/disc.fib.}) \qquad (\text{initial/disc.opfib.})$$

Recall the following

Fact

For a functor $\mathbb{A} \xrightarrow{F} \mathbb{B}$, T.F.A.E.

- (i) F is final;
- (ii) for every object b of $\mathbb{B}$, the comma category (b/F) is non-empty and connected;
- (iii) $F \perp \mathcal{D}$;
- (iv) for all $G: \mathbb{B} \rightarrow \mathbb{E}$, $\text{colim } G \circ F \simeq \text{colim } G$.

Distributors

A distributor [Bénabou 1973] $\mathbb{A} \xrightarrow{S} \mathbb{B}$ is a set valued functor

$$S: \mathbb{B}^{\text{op}} \times \mathbb{A} \longrightarrow \mathbf{Set} .$$

Distributors from $\mathbb{A}$ to $\mathbb{B}$ form the category $\mathbf{Dist}(\mathbb{A}, \mathbb{B})$.

Given distributors $\mathbb{C} \xleftarrow{T} \mathbb{B} \xleftarrow{S} \mathbb{A}$ define their composition

$$T \otimes S: \mathbb{C}^{\text{op}} \times \mathbb{A} \longrightarrow \mathbf{Set}$$

where $T \otimes S(c, a)$ is a suitable quotient of $\coprod_{b \in \mathbb{B}} T(c, b) \times S(b, a)$.

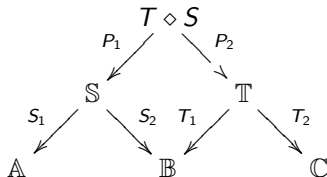
Distributor composition is associative and gives the bicategory $\mathbf{Dist}$.

Distributors can be described by discrete two-sided fibrations

$$S \longrightarrow \mathbb{A} \times \mathbb{B}$$

How can we relate distributors composition with their associated spans?

Given distributors S and T , their composition as spans (=discrete two-sided fibrations) is obtained by the pullback



The resulting span $(S_1 P_1, T_2 P_2)$ is no longer a discrete two-sided fibration in general, but still **it is a two-sided fibration**

Therefore, we are to study the *discrete object* reflection...

$$\mathcal{R}': \mathbf{Fib}(\mathbb{A}, \mathbb{C}) \longrightarrow \mathbf{DFib}(\mathbb{A}, \mathbb{C}) = \mathbf{Dist}(\mathbb{A}, \mathbb{C})$$

... given by the identree/coidentifier factorization: for $X \in \mathbf{Fib}(\mathbb{A}, \mathbb{C})$

$$\mathbb{K}(X) \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \kappa \\ \xrightarrow{\quad} \end{array} X \xrightarrow{Q_X} \mathcal{R}'(X) \xrightarrow{\mathcal{R}'(X)} \mathbb{A} \times \mathbb{C}$$

X

The fibred viewpoint

The discrete reflection of two-sided fibrations can be interpreted in terms of the comprehensive factorization in (op)fibrations.

Proposition

For $X \in \mathbf{Fib}(\mathbb{A}, \mathbb{C})$, $X = \mathcal{R}'(X) \circ Q_X$ gives

- the (final/discrete fibration) factorization in $\mathbf{opFib}(\mathbb{A})$

$$\begin{array}{ccccc} X & \xrightarrow{Q_X} & \mathcal{R}'(X) & \xrightarrow{\mathcal{R}'(X)} & \mathbb{A} \times \mathbb{C} \\ & \searrow & \downarrow & \swarrow \pi_1 & \\ & & \mathbb{A} & & \end{array}$$

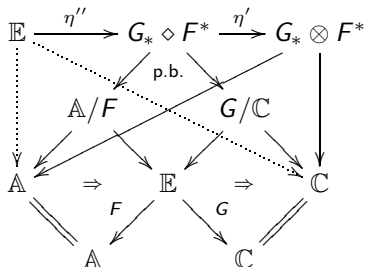
- the (initial/discrete opfibration) factorization in $\mathbf{Fib}(\mathbb{C})$

This is based on a fiberwise characterization of (op)fibrations in $\mathbf{Cat}/\mathbb{A}$ and in $\mathbf{Fib}(\mathbb{A})$ [Cigoli, Mantovani, M., Vitale 2018]

Proposition

The local reflections of spans into distributors factorize through two-sided fibrations

$$\mathbf{Dist}(\mathbb{A}, \mathbb{C}) \xrightleftharpoons[\mathcal{I}']{\mathcal{R}' \atop \perp} \mathbf{Fib}(\mathbb{A}, \mathbb{C}) \xrightleftharpoons[\mathcal{I}''']{\mathcal{R}'' \atop \perp} \mathbf{Span}(\mathbb{A}, \mathbb{C})$$



$$G_* \diamond F^* = \mathcal{R}''(F, G), \quad G_* \otimes F^* = \mathcal{R}'(G_* \diamond F^*)$$

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Internal distributors

Let $\mathcal{E}$ be Barr-exact (here and forever).

Notation

Cat($\mathcal{E}$) is the category of internal categories and internal functors.

Gpd($\mathcal{E}$) is the category of internal groupoids and internal functors.

$$\begin{array}{ccc} C_1 & \xrightarrow{F_1} & D_1 \\ \downarrow d \uparrow c & & \downarrow d \uparrow c \\ C_0 & \xrightarrow{F_0} & D_0 \end{array}$$

SpanGpd($\mathcal{E}$) is the bicategory of spans between internal groupoids.

DistGpd($\mathcal{E}$) is the bicategory of distributors between internal groupoids.

Many features of the bicategory **DistGpd**($\mathcal{E}$) can be described in set-theoretical terms, i.e. in **DistGpd**(**Set**).

Here we recall the ones that are relevant to our purposes.

Internal distributors

Definition

An internal distributor $A \xrightarrow[S]{\quad} B$ is given by

- a span $A_0 \longleftarrow S \longrightarrow B_0$ in $\mathcal{E}$
- a left action $A_1 \times_{A_0} S \xrightarrow{\lambda} S$
- a right action $S \times_{B_0} B_1 \xrightarrow{\rho} S$

which are associative, unital and compatible.

Given distributors $A \xrightarrow[S]{\quad} B \xrightarrow[T]{\quad} C$, their composition is given by the coequalizer in $\mathcal{E}$:

$$\begin{array}{ccccc}
 S \times_{B_0} B_1 \times_{B_0} T & \xrightarrow{\rho_S \times 1} & S \times_{B_0} T & \xrightarrow{Q} & T \otimes S \\
 & \xrightarrow{1 \times \lambda_T} & & & \downarrow \\
 & & & & A_0 \times C_0
 \end{array}$$

Facts about (internal) groupoids

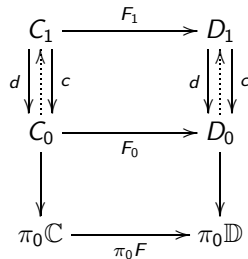
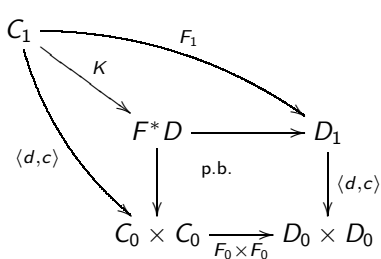
In the category of groupoids in **Set**:

- (i) $\mathcal{D} = \mathcal{D}^{op}$, i.e. a functor is a discrete fibration iff it is a discrete opfibration;
- (ii) $\mathbb{E} \longrightarrow \mathbb{A} \times \mathbb{B}$ is a two-sided discrete fibration iff it is a discrete fibration
- (iii) $\mathcal{F} = \mathcal{F}^{op}$, i.e. a functor is final iff it is initial;
- (iv) a functor is final iff it is full and essentially surjective.

For internal groupoids in $\mathcal{E}$:

- (i) and (ii) come for free;
- (iii) holds since orthogonality is a representable notion;
- (iv) is due to [Cigoli, 2017].

Facts about (internal) groupoids

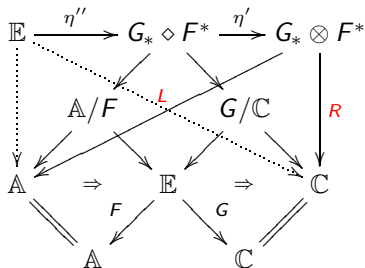


(v) a functor $F = (F_0, F_1)$ is full and essentially surjective iff K is a regular epi and $\pi_0 F$ is an iso.

The local reflections

Proposition

Fixed two groupoids $\mathbb{A}$ and $\mathbb{B}$, the comprehensive factorization gives a (final) reflection $\mathcal{R} = \mathcal{R}' \circ \mathcal{R}''$ to the inclusion of distributors into spans

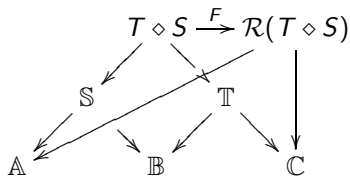


- η' is at the same time initial and final – since it is a coidentifier.
- In the case of groupoids, η'' is an equivalence.
- In the case of groupoids, $\langle L, R \rangle$ is discrete fibration.

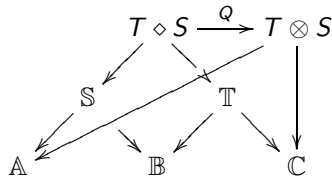
Lemma

For given internal groupoids $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$, the following commutes:

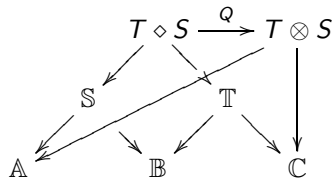
$$\begin{array}{ccc}
 \mathbf{Dist}(\mathbb{A}, \mathbb{B}) \times \mathbf{Dist}(\mathbb{B}, \mathbb{C}) & \xrightarrow{\otimes} & \mathbf{Dist}(\mathbb{A}, \mathbb{C}) \\
 \mathcal{I} \times \mathcal{I} \downarrow & & \uparrow \mathcal{R} \\
 \mathbf{Span}(\mathbb{A}, \mathbb{B}) \times \mathbf{Span}(\mathbb{B}, \mathbb{C}) & \xrightarrow{\diamond} & \mathbf{Span}(\mathbb{A}, \mathbb{C})
 \end{array}$$



F is a coidentifier $\Rightarrow$ final



only need to prove Q is final!



$$\begin{array}{ccccc}
 (T \diamond S)_1 & \xrightarrow{Q_1} & (T \otimes S)_0 \times_0 (A_1 \times C_1) & \xrightarrow{\quad} & A_1 \times C_1 \\
 \updownarrow & & \updownarrow & \text{p.b.} & \updownarrow \\
 (T \diamond S)_0 & \xrightarrow{Q_0} & (T \otimes S)_0 & \xrightarrow{\quad} & A_0 \times C_0
 \end{array}$$

Lemma

For given internal groupoids $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$, the following commutes:

$$\begin{array}{ccc}
 \mathbf{Dist}(\mathbb{A}, \mathbb{B}) \times \mathbf{Dist}(\mathbb{B}, \mathbb{C}) & \xrightarrow{\otimes} & \mathbf{Dist}(\mathbb{A}, \mathbb{C}) \\
 \mathcal{I} \times \mathcal{I} \downarrow & & \uparrow \mathcal{R} \\
 \mathbf{Span}(\mathbb{A}, \mathbb{B}) \times \mathbf{Span}(\mathbb{B}, \mathbb{C}) & \xrightarrow{\diamond} & \mathbf{Span}(\mathbb{A}, \mathbb{C})
 \end{array}$$

Theorem

$$\mathbf{DistGpd}(\mathcal{E}) = \mathbf{Rel}(\mathbf{Gpd}(\mathcal{E}); (\mathcal{F}, \mathcal{D}))$$

i.e. the bicategory of distributors between groupoids in $\mathcal{E}$ is the bicategory of relations in $\mathbf{Gpd}(\mathcal{E})$ relative to the (final/disc.fib.) f.s.



J. BÉNABOU, Les distributeurs, *Université catholique de Louvain, I.M.P.A., rapport 33* (1973).



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Thank you for your attention!