Fibred and internal aspects of distributors composition

Giuseppe Metere

P.S.S.L. 103
Brno 2018/04/7-8

July 6, 2026

- Introduction
- Fibred aspects of distributors composition
- The internal case

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Relations in categories

Let $\mathcal{C}$ be a finitely complete category, endowed with a $(\mathcal{E}, \mathcal{M})$-orthogonal factorization system. A *relation* between objects A and B

$$A \overset{S}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B$$

is an $\mathcal{M}$-subobject S of $A \times B$.

Relations between A and B give the category $\mathbf{Rel}(A, B)$

Every relation $S \subset A \times B$ determines a span: $A \longleftarrow S \longrightarrow B$, and the local embedding of relations into spans admits a reflection given by the $(\mathcal{E}, \mathcal{M})$-factorization:

$$\begin{tikzcd} \mathbf{Rel}(A, B) \arrow[r, shift right, "i_{A,B}"'] \arrow[r, phantom, "\perp"] & \mathbf{Span}(A, B) \arrow[l, shift right, "r_{A,B}"'] \end{tikzcd}$$

Relations in categories

Relations compose:

$$
\begin{tikzcd}
A \arrow[r, "S", "|"' marking] & B \arrow[r, "T", "|"' marking] & C
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & T \diamond S \arrow[r, "r_{A,C}"] \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\text{p.b.}"] & & T \circ S \arrow[ddllll] \arrow[dd] \\
& S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\
A & & B & & C
\end{tikzcd}
$$

but the composition of relations is not associative, in general.

When the composition of relations in $\mathcal{C}$ is associative, relations form the bicategory

$$\mathbf{Rel}(\mathcal{C}) = \mathbf{Rel}(\mathcal{C}, (\mathcal{E}, \mathcal{M}))$$

of relations relative to the $(\mathcal{E}, \mathcal{M})$-factorization system.

Example: For $\mathcal{C} = \mathbf{Set}$, $\mathcal{E} =$ surjections and $\mathcal{M} =$ injections, one obtains the usual notion of relation between sets.

Relations in categories

Theorem (Klein 1970, Kelly 1990)

If $\mathcal{M} \subset$ Mono, then the composition of relation is associative iff the factorization system $(\mathcal{E}, \mathcal{M})$ is stable.

However, there are relevant examples of bicategories of relations relative to factorization system where the hypothesis $\mathcal{M} \subset$ Mono does not hold, and the class $\mathcal{E}$ is not stable under pullbacks.

Theorem (M. 2016, Distributors = Relators)

$$\mathbf{DistGpd} = \mathbf{Rel}(\mathbf{Gpd}, (\mathcal{F}, \mathcal{D}))$$

where $\mathcal{F}$ = final functors, and $\mathcal{D}$ = discrete fibrations.

Plan of the talk

1. **Internal groupoids:** if $\mathcal{E}$ is Barr-exact,

$$\mathbf{DistGpd}(\mathcal{E}) = \mathbf{Rel}(\mathbf{Gpd}(\mathcal{E}), (\mathcal{F}, \mathcal{D}))$$

2. **Categories:** distributor composition from a fibrational point of view.

...but we shall start from number 2.

Overview

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Definition

A two-sided fibration from $\mathbb{A}$ to $\mathbb{B}$ is a span of categories and functors

$$\mathbb{A} \xleftarrow{Q} \mathbb{E} \xrightarrow{P} \mathbb{B}$$

such that:

(i) each arrow $b \xrightarrow{\beta} P(e')$ in $\mathbb{B}$ has a P-cartesian Q-vertical lift at e',

(ii) each arrow $Q(e) \xrightarrow{\alpha} a$ in $\mathbb{A}$ has a Q-opcartesian P-vertical lift at e,

(iii) for every cartesian opcartesian composite $\beta^* e \longrightarrow e \longrightarrow \alpha_* e$, the canonical comparison $\alpha_* \beta^* e \longrightarrow \beta^* \alpha_* e$ is an isomorphism.

It is called *discrete* when it has discrete fibers (over $\mathbb{A} \times \mathbb{B}$).

Fib$(\mathbb{A}, \mathbb{B})$ *is the category of two-sided fibrations from* $\mathbb{A}$ *to* $\mathbb{B}$.

$$\textbf{opFib}(\mathbb{A}) = \textbf{Fib}(\mathbb{A}, 1) \qquad \textbf{Fib}(\mathbb{B}) = \textbf{Fib}(1, \mathbb{B})$$

Two factorization systems in **Cat**

The comprehensive factorization of a functor [Street and Walters 1973].

$(\mathcal{D}, \mathcal{F})$ $(\mathcal{D}^{op}, \mathcal{F}^{op})$

(final/disc.fib.) (initial/disc.opfib.)

Recall the following

Fact

For a functor $\mathbb{A} \xrightarrow{F} \mathbb{B}$, *T.F.A.E.*

(i) *F is final;*

(ii) *for every object b of $\mathbb{B}$, the comma category (b/F) is non-empty and connected;*

(iii) *$F \perp \mathcal{D}$;*

(iv) *for all $G : \mathbb{B} \to \mathbb{E}$, colim $G \circ F \simeq$ colim G.*

Distributors

A distributor [Bénabou 1973] $\mathbb{A} \overset{S}{\longrightarrow\!\!\!\!\!\!\!+\ \ } \mathbb{B}$ is a set valued functor

$$S\colon \mathbb{B}^{\mathrm{op}} \times \mathbb{A} \longrightarrow \mathbf{Set}\ .$$

Distributors from $\mathbb{A}$ to $\mathbb{B}$ form the category $\mathbf{Dist}(\mathbb{A}, \mathbb{B})$.

Given distributors $\mathbb{C} \overset{T}{\longleftarrow\!\!\!\!\!\!\!+\ \ } \mathbb{B} \overset{S}{\longleftarrow\!\!\!\!\!\!\!+\ \ } \mathbb{A}$ define their composition

$$T \otimes S\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{A} \longrightarrow \mathbf{Set}$$

where $T \otimes S\,(c, a)$ is a suitable quotient of $\coprod_{b \in \mathbb{B}} T(c, b) \times S(b, a)$.

Distributor composition is associative and gives the bicategory **Dist**.

Distributors can be described by discrete two-sided fibrations

$$\mathbb{S} \longrightarrow \mathbb{A} \times \mathbb{B}$$

How can we relate distributors composition with their associated spans?

Given distributors S and T, their composition as spans (=discrete two-sided fibrations) is obtained by the pullback

$$
\begin{tikzcd}
 & & T \diamond S \arrow[dl, "P_1"'] \arrow[dr, "P_2"] & & \\
 & \mathbb{S} \arrow[dl, "S_1"'] \arrow[dr, "S_2"] & & \mathbb{T} \arrow[dl, "T_1"'] \arrow[dr, "T_2"] & \\
\mathbb{A} & & \mathbb{B} & & \mathbb{C}
\end{tikzcd}
$$

> *The resulting span $(S_1 P_1, T_2 P_2)$ is no longer a discrete two-sided fibration in general, but still* **it is a two-sided fibration**

Therefore, we are to study the *discrete object* reflection...

$$\mathcal{R}' \colon \mathbf{Fib}(\mathbb{A}, \mathbb{C}) \longrightarrow \mathbf{DFib}(\mathbb{A}, \mathbb{C}) = \mathbf{Dist}(\mathbb{A}, \mathbb{C})$$

...given by the identee/coidentifier factorization: for $X \in \mathbf{Fib}(\mathbb{A}, \mathbb{C})$

$$
\begin{tikzcd}
\mathbb{K}(X) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\kappa"] & \mathbb{X} \arrow[r, "\mathcal{Q}_X"] \arrow[rr, bend right, "X"'] & \mathcal{R}'(\mathbb{X}) \arrow[r, "\mathcal{R}'(X)"] & \mathbb{A} \times \mathbb{C}
\end{tikzcd}
$$

The fibred viewpoint

The discrete reflection of two-sided fibrations can be interpreted in terms of the comprehensive factorization in (op)fibrations.

Proposition

For $X \in \mathbf{Fib}(\mathbb{A},\mathbb{C})$, $X = \mathcal{R}'(X) \circ \mathcal{Q}_X$ gives

- *the (final/discrete fibration) factorization in* $\mathbf{opFib}(\mathbb{A})$

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\mathcal{Q}_X"] \arrow[dr] & \mathcal{R}'(\mathbb{X}) \arrow[r, "\mathcal{R}'(X)"] \arrow[d] & \mathbb{A} \times \mathbb{C} \arrow[dl, "\pi_1"] \\ & \mathbb{A} & \end{tikzcd}$$

- *the (initial/discrete opfibration) factorization in* $\mathbf{Fib}(\mathbb{C})$

This is based on a fiberwise characterization of (op)fibrations in $\mathbf{Cat}/\mathbb{A}$ and in $\mathbf{Fib}(\mathbb{A})$ [Cigoli, Mantovani, M., Vitale 2018]

Proposition

The local reflections of spans into distributors factorize through two-sided fibrations

$$\begin{tikzcd} \mathbf{Dist}(\mathbb{A},\mathbb{C}) \arrow[r, shift right, "\mathcal{I}'"'] \arrow[r, phantom, "\perp"] & \mathbf{Fib}(\mathbb{A},\mathbb{C}) \arrow[l, shift right, "\mathcal{R}'"'] \arrow[r, shift right, "\mathcal{I}''"'] \arrow[r, phantom, "\perp"] & \mathbf{Span}(\mathbb{A},\mathbb{C}) \arrow[l, shift right, "\mathcal{R}''"'] \end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, "\eta''"] \arrow[dd, dotted] \arrow[ddrrrr, dotted] & & G_* \diamond F^* \arrow[rr, "\eta'"] \arrow[dl] \arrow[dr] \arrow[d, phantom, "\text{p.b.}"] & & G_* \otimes F^* \arrow[ddllll] \arrow[dd] \\
& \mathbb{A}/F \arrow[dl] \arrow[dr] & {} & G/\mathbb{C} \arrow[dl] \arrow[dr] & \\
\mathbb{A} \arrow[r, phantom, "\Rightarrow"] \arrow[dr, equal] & {} & \mathbb{E} \arrow[dl, "F"'] \arrow[dr, "G"] \arrow[r, phantom, "\Rightarrow"] & {} & \mathbb{C} \arrow[dl, equal] \\
& \mathbb{A} & & \mathbb{C} &
\end{tikzcd}$$

$$G_* \diamond F^* = \mathcal{R}''(F,G)\,, \qquad G_* \otimes F^* = \mathcal{R}'(G_* \diamond F^*)$$

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Internal distributors

Let $\mathcal{E}$ be Barr-exact (here and forever).

Notation

$\mathbf{Cat}(\mathcal{E})$ *is the category of internal categories and internal functors.*
$\mathbf{Gpd}(\mathcal{E})$ *is the category of internal groupoids and internal functors.*

$$
\begin{tikzcd}
C_1 \arrow[r, "F_1"] \arrow[d, shift right=2ex, "d"'] \arrow[d, shift left=2ex, "c"] & D_1 \arrow[d, shift right=2ex, "d"'] \arrow[d, shift left=2ex, "c"] \\
C_0 \arrow[u] \arrow[r, "F_0"'] & D_0 \arrow[u]
\end{tikzcd}
$$

$\mathbf{SpanGpd}(\mathcal{E})$ *is the bicategory of spans between internal groupoids.*
$\mathbf{DistGpd}(\mathcal{E})$ *is the bicategory of distributors between internal groupoids.*

Many features of the bicategory $\mathbf{DistGpd}(\mathcal{E})$ can be described in set-theoretical terms, i.e. in $\mathbf{DistGpd}(\mathbf{Set})$.
Here we recall the ones that are relevant to our purposes.

Internal distributors

Definition

An internal distributor $\mathbb{A} \overset{S}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} \mathbb{B}$ is given by

- a span $A_0 \longleftarrow S \longrightarrow B_0$ in $\mathcal{E}$
- a left action $A_1 \times_{A_0} S \xrightarrow{\lambda} S$
- a right action $S \times_{B_0} B_1 \xrightarrow{\rho} S$

which are associative, unital and compatible.

Given distributors $\mathbb{A} \overset{S}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} \mathbb{B} \overset{T}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} \mathbb{C}$, their composition is given by the coequalizer in $\mathcal{E}$:

$$
\begin{tikzcd}
S \times_{B_0} B_1 \times_{B_0} T \arrow[r, shift left, "\rho_S \times 1"] \arrow[r, shift right, "1 \times \lambda_T"'] & S \times_{B_0} T \arrow[r, two heads, "Q"] \arrow[dr] & T \otimes S \arrow[d] \\
& & A_0 \times C_0
\end{tikzcd}
$$

Facts about (internal) groupoids

In the category of groupoids in **Set**:

(i) $\mathcal{D} = \mathcal{D}^{op}$, *i.e. a functor is a discrete fibration iff it is a discrete opfibration;*

(ii) $\mathbb{E} \longrightarrow \mathbb{A} \times \mathbb{B}$ *is a two-sided discrete fibration iff it is a discrete fibration*

(iii) $\mathcal{F} = \mathcal{F}^{op}$, *i.e. a functor is final iff it is initial;*

(iv) *a functor is final iff it is full and essentially surjective.*

For internal groupoids in $\mathcal{E}$:

(i) and (ii) come for free;
(iii) holds since orthogonality is a representable notion;
(iv) is due to [Cigoli, 2017].

Facts about (internal) groupoids

$$
\begin{tikzcd}
C_1 \arrow[dr, "K"] \arrow[drr, bend left, "F_1"] \arrow[ddr, bend right, "{\langle d,c \rangle}"'] & & \\
& F^*D \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & D_1 \arrow[d, "{\langle d,c \rangle}"] \\
& C_0 \times C_0 \arrow[r, "F_0 \times F_0"'] & D_0 \times D_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[r, "F_1"] \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] & D_1 \arrow[d, shift right=2, "d"'] \arrow[d, shift left=2, "c"] \\
C_0 \arrow[u, dotted] \arrow[r, "F_0"'] \arrow[d] & D_0 \arrow[u, dotted] \arrow[d] \\
\pi_0\mathbb{C} \arrow[r, "\pi_0 F"'] & \pi_0\mathbb{D}
\end{tikzcd}
$$

(v) *a functor $F = (F_0, F_1)$ is full and essentially surjective iff K is a regular epi and $\pi_0 F$ is an iso.*

The local reflections

Proposition

Fixed two groupoids $\mathbb{A}$ and $\mathbb{B}$, the comprehensive factorization gives a (final) reflection $\mathcal{R} = \mathcal{R}' \circ \mathcal{R}''$ to the inclusion of distributors into spans

$$\begin{tikzcd}
\mathbb{E} \arrow[rr, "\eta''"] \arrow[dd, dotted] \arrow[ddrrrr, dotted] & & G_* \diamond F^* \arrow[rr, "\eta'"] \arrow[dl] \arrow[dr] & & G_* \otimes F^* \arrow[ddllll, "L"] \arrow[dd, "R"] \\
& \mathbb{A}/F \arrow[dl] \arrow[dr] & & G/\mathbb{C} \arrow[dl] \arrow[dr] & \\
\mathbb{A} \arrow[rr, phantom, "\Rightarrow"] \arrow[dr, equal] & & \mathbb{E} \arrow[rr, phantom, "\Rightarrow"] \arrow[dl, "F"'] \arrow[dr, "G"] & & \mathbb{C} \arrow[dl, equal] \\
& \mathbb{A} & & \mathbb{C} &
\end{tikzcd}$$

- η' is at the same time initial and final – since it is a coidentifier.
- In the case of groupoids, η'' is an equivalence.
- In the case of groupoids, $\langle L, R \rangle$ is discrete fibration.

Lemma

For given internal groupoids $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$, the following commutes:

$$\begin{tikzcd}
\mathbf{Dist}(\mathbb{A},\mathbb{B}) \times \mathbf{Dist}(\mathbb{B},\mathbb{C}) \arrow[r, "\otimes"] \arrow[d, "\mathcal{I}\times\mathcal{I}"'] & \mathbf{Dist}(\mathbb{A},\mathbb{B}) \\
\mathbf{Span}(\mathbb{A},\mathbb{B}) \times \mathbf{Span}(\mathbb{B},\mathbb{C}) \arrow[r, "\diamond"'] & \mathbf{Span}(\mathbb{A},\mathbb{B}) \arrow[u, "\mathcal{R}"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & & T\diamond S \arrow[rr, "F"] \arrow[dl] \arrow[dr] & & \mathcal{R}(T\diamond S) \arrow[ddllll] \arrow[dd] \\
 & \mathbb{S} \arrow[dl] \arrow[dr] & & \mathbb{T} \arrow[dl] \arrow[dr] & \\
\mathbb{A} & & \mathbb{B} & & \mathbb{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & T\diamond S \arrow[rr, "Q"] \arrow[dl] \arrow[dr] & & T\otimes S \arrow[ddllll] \arrow[dd] \\
 & \mathbb{S} \arrow[dl] \arrow[dr] & & \mathbb{T} \arrow[dl] \arrow[dr] & \\
\mathbb{A} & & \mathbb{B} & & \mathbb{C}
\end{tikzcd}$$

F is a coidentifier $\Rightarrow$ final

only need to prove Q is final!

$$
\begin{tikzcd}
 & & T \diamond S \arrow[rr, "Q"] \arrow[dl] \arrow[dr] & & T \otimes S \arrow[ddllll] \arrow[dd] \\
 & \mathbb{S} \arrow[dl] \arrow[dr] & & \mathbb{T} \arrow[dl] \arrow[dr] & \\
\mathbb{A} & & \mathbb{B} & & \mathbb{C}
\end{tikzcd}
$$

$$
\begin{tikzcd}
(T \diamond S)_1 \arrow[r, dashed, "Q_1"] \arrow[d, shift right=2] \arrow[d, shift left=2] & (T \otimes S)_0 \times_0 (A_1 \times C_1) \arrow[r] \arrow[d, shift right=2] \arrow[d, shift left=2] \arrow[dr, phantom, "\text{p.b.}"] & A_1 \times C_1 \arrow[d, shift right=2] \arrow[d, shift left=2] \\
(T \diamond S)_0 \arrow[u] \arrow[r, two heads, "Q_0"'] & (T \otimes S)_0 \arrow[u] \arrow[r] & A_0 \times C_0 \arrow[u]
\end{tikzcd}
$$

Lemma

For given internal groupoids $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$, the following commutes:

$$\begin{tikzcd} \mathbf{Dist}(\mathbb{A},\mathbb{B}) \times \mathbf{Dist}(\mathbb{B},\mathbb{C}) \arrow[r, "\otimes"] \arrow[d, "\mathcal{I}\times\mathcal{I}"'] & \mathbf{Dist}(\mathbb{A},\mathbb{B}) \\ \mathbf{Span}(\mathbb{A},\mathbb{B}) \times \mathbf{Span}(\mathbb{B},\mathbb{C}) \arrow[r, "\diamond"'] & \mathbf{Span}(\mathbb{A},\mathbb{B}) \arrow[u, "\mathcal{R}"'] \end{tikzcd}$$

Theorem

$$\mathbf{DistGpd}(\mathcal{E}) = \mathbf{Rel}(\mathbf{Gpd}(\mathcal{E}); (\mathcal{F},\mathcal{D}))$$

i.e. the bicategory of distributors between groupoids in $\mathcal{E}$ is the bicategory of relations in $\mathbf{Gpd}(\mathcal{E})$ relative to the (final/disc.fib.) f.s.

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Thank you for your attention!