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Groupal pseudofunctors and cohomology 2-groups

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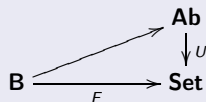


[†] *From an ongoing joint project with Alan S. Cigoli and Sandra Mantovani*

Intro

Fact

Every **Set**-valued functor F , from an abelian category $\mathbf{B}$, that preserves finite products, factors through the category $\mathbf{Ab}$ of abelian groups.



Example: [Yoneda, 1960] defines the cohomology functors

$$\mathrm{Ext}^n(A, -): \mathbf{B} \longrightarrow \mathbf{Set}$$

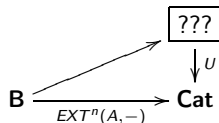
$$B \mapsto \mathrm{Ext}^n(A, B) = \{[0 \rightarrow B \rightarrow E_n \rightarrow \cdots \rightarrow E_1 \rightarrow A \rightarrow 0]_{\sim}\}$$

where $\sim$ is the **connectedness** relation.

What if we do not take the quotient on the connectedness relation?

We obtain a pseudofunctor $\mathrm{EXT}^n(A, -): \mathbf{B} \longrightarrow \mathbf{Cat}$.

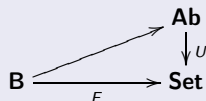
Q: under what conditions on this pseudofunctor does $\mathbf{B}$ induce an additional structure on each category $\mathrm{EXT}^n(A, B)$?



Intro

Fact

Every **Set**-valued functor F , from an abelian category $\mathbf{B}$, that preserves finite products factors through the category $\mathbf{Ab}$ of abelian groups.



This fact is a combination of two elementary observations:

- (i) for $\mathbf{B}$ abelian, the forgetful functor $U: \mathbf{Ab}(\mathbf{B}) \rightarrow \mathbf{B}$ is an isomorphism
- (ii) a **product preserving functor** takes (abelian) groups to (abelian) groups the latter being a special instance of
- (iii) a **monoidal functor** takes monoids to monoids

The aim of my talk is to explain how this whole story can be told in dimension 2

Since (i) will not be affected by our dimensional shift, we start from a well-known 2-categorical version of (iii), and then that of (ii).

First step: 2-categorical version of (iii)

Fact (Day Street 1997)

lax (2-)monoidal pseudofunctors take pseudomonoids to pseudomonoids:

$$\boxed{M \otimes M \xrightarrow{*} M} \mapsto \boxed{F(M) \otimes F(M) \xrightarrow{R^{M,M}} F(M \otimes M) \xrightarrow{F(*)} F(M)}$$

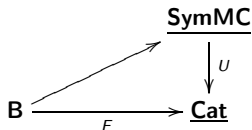
where a **pseudomonoid** in a monoidal 2-category is a **monoid object** $(M, *, e_M)$, s.t. monoid axioms hold up to coherent isomorphisms.

If $\mathbf{B}$ has finite products, one can consider a pseudofunctor $F: \mathbf{B} \rightarrow \underline{\mathbf{Cat}}$, lax monoidal w.r.t the cartesian structures. Then F lifts:

$$\begin{array}{ccc} \mathbf{psMon}(\mathbf{B}) = \mathbf{Mon}(\mathbf{B}) & \dashrightarrow & \mathbf{psMon}(\underline{\mathbf{Cat}}) = \mathbf{MonCat} \\ \downarrow U & & \downarrow U \\ \mathbf{B} & \xrightarrow{F} & \underline{\mathbf{Cat}} \end{array}$$

First step: 2-categorical version of (iii)

If $\mathbf{B}$ is semiadditive ($\times = +$), the forgetful functor $U: \mathbf{CMon}(\mathbf{B}) \rightarrow \mathbf{B}$ is an iso, and F factors through the 2-category of symmetric monoidal categories:



Example

Consider the lax monoidal pseudofunctor

$$\text{Sub}(-): (\mathbf{Ab}, \oplus, 0) \rightarrow (\underline{\mathbf{Cat}}, \times, \mathbf{I})$$

that assigns to every abelian group A the poset $\text{Sub}(A)$ of its subgroups. Every A induces a symmetric monoidal structure on $\text{Sub}(A)$, with tensor product given by the join of subgroups.

Second step: pseudogroups

Now we look for a result *à la* Day-Street, which would state if/when a monoidal pseudofunctor $F: \mathbf{B} \rightarrow \mathbf{Cat}$ takes internal groups to...

to what?

Definition (Baez Lauda, 2004)

A **pseudogroup** (*weak 2-group* in [*loc.cit.*]) in a 2-category $\mathbf{K}$ with fin. prod. is a pseudomonoid $(G, \otimes, I)$ endowed with a 1-cell $(-)^*: G \rightarrow G$ and iso 2-cells with components $X \otimes X^* \rightarrow I$, $I \rightarrow X^* \otimes X$ satisfying coherence conditions.

In $\mathbf{Cat}$, they are monoidal groupoids s.t. every object is $\otimes$ -pseudoinvertible.

Why monoidal groupoids, and not just monoidal categories?

Fact: All the 1-cells of a pseudogroup $\mathbf{G}$ in $\mathbf{Cat}$ are invertible w.r.t. composition, i.e. $\mathbf{G}$ is a groupoid.

[Baez Lauda, 2004]

Reason: want $(-)^*: \mathbf{G} \rightarrow \mathbf{G}$ to be internal, i.e. a *covariant* functor. Requiring just pseudoinvertible objects produces a contravariant functor.

Example: preordered groups.

Third step: groupal pseudofunctors

Recall that an object C of a bicategory with products is a **cartesian object** if the 1-cells $\Delta: C \rightarrow C \times C$ and $C \rightarrow 1$ have right adjoint.

Let **psFun** be the 2-category (with pointwise defined fin. prod.) of pseudofunctors $\bar{F}: \mathbf{B} \rightarrow \underline{\mathbf{Cat}}$, pseudocommutative triangles over $\underline{\mathbf{Cat}}$, and their morphisms.

Definition

A cartesian object in **psFun** is called **lax cartesian monoidal pseudofunctor**.

In this the case, $\mathbf{B}$ has finite products, and F is lax monoidal

$$(F, R, R^1, \dots): (\mathbf{B}, \times, 1) \rightarrow (\underline{\mathbf{Cat}}, \times, \mathbf{I})$$

where the components of the structural pseudonatural transformations are:

$$R^{A,B}: F(A) \times F(B) \rightarrow F(A \times B) \quad \text{and} \quad R^1: \mathbf{I} \rightarrow F(1)$$

and we won't mention the coherent modifications.

We'll see that $R^{A,B}(X, Y)$ and $R^1(\star)$ can be interpreted as binary and 0-ary products.

Third step: groupal pseudofunctors

Definition

A lax cartesian monoidal pseudofunctor $F: (\mathbf{B}, \times, I) \rightarrow (\underline{\mathbf{Cat}}, \times, I)$ is **groupal** if it lifts to a pseudofunctor $\hat{F}$:

$$\begin{array}{ccc} \mathbf{Gp}(\mathbf{B}) & \xrightarrow{\hat{F}} & \underline{2\mathbf{Gp}} \\ U \downarrow & & \downarrow U \\ \mathbf{B} & \xrightarrow{F} & \underline{\mathbf{Cat}} \end{array}$$

Fact

In this case, F also lifts to abelian groups:

$$\begin{array}{ccc} \mathbf{Ab}(\mathbf{B}) & \xrightarrow{\hat{\hat{F}}} & \underline{\mathbf{Sym2Gp}} \\ U \downarrow & & \downarrow U \\ \mathbf{B} & \xrightarrow{F} & \underline{\mathbf{Cat}} \end{array}$$

The lax cartesian monoidal pseudofunctor of subobjects:

$$\mathbf{Sub}(-): (\mathbf{Ab}, \oplus, 0) \rightarrow (\underline{\mathbf{Cat}}, \times, I)$$

shows that a result *à la* Day-Street **does not hold** in general.

Counterexample: For $1 \neq A \in \mathbf{Ab}$, the preorder $\mathbf{Sub}(A)$ is not a groupoid,
 $\Rightarrow \mathbf{Sub}(A)$ cannot support any 2-group structure.

AIM: characterize groupal pseudofunctors.

Proposition (Cigoli, Mantovani, M., 2023)

Let $\mathbf{B}$ be a category with fin. prod., and $F: \mathbf{B} \rightarrow \underline{\mathbf{Cat}}$ be a pseudofunctor. Then F is canonically endowed with an **oplax symmetric monoidal structure**

$$(F, L, L^1, \dots): (\mathbf{B}, \times, I) \longrightarrow (\underline{\mathbf{Cat}}, \times, I)$$

$$\begin{array}{ccc}
 \mathbf{B} \times \mathbf{B} & \xrightarrow{\times} & \mathbf{B} \\
 F \times F \downarrow & \swarrow L & \downarrow F \\
 \underline{\mathbf{Cat}} \times \underline{\mathbf{Cat}} & \xrightarrow{\times} & \underline{\mathbf{Cat}}
 \end{array}
 \quad
 \begin{array}{ccc}
 I & \xrightarrow{I} & \mathbf{B} \\
 & \searrow L^1 & \downarrow F \\
 & I & \underline{\mathbf{Cat}}
 \end{array}
 \quad
 \begin{array}{l}
 L^{A,B}: F(A \times B) \rightarrow F(A) \times F(B) \\
 Z \mapsto (F(\pi_1)(Z), F(\pi_2)(Z)) \\
 \\
 L^1: F(I) \rightarrow I \\
 J \mapsto *
 \end{array}$$

Lemma (Cigoli, Mantovani, M.)

Let $\mathbf{B}$ be a cat. with fin. prod., and $F: \mathbf{B} \rightarrow \underline{\mathbf{Cat}}$ be a pseudofunctor. TFAE:

- F is lax cartesian monoidal, i.e. F is a cartesian object in psFun
- L and L^1 have a right adjoints R and R^1

Theorem (Cigoli, Mantovani, M., 2023)

Let $\mathbf{B}$ be a category with finite products, and $F: \mathbf{B} \rightarrow \underline{\mathbf{Cat}}$ a lax cartesian monoidal pseudofunctor. TFAE:

- F is **groupal**
- For every A in $\mathbf{B}$ supporting an internal group structure, the maps

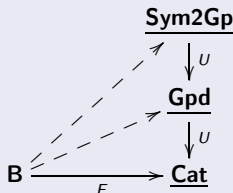
$$\eta^1: id_{F(I)} \Rightarrow R^1 \circ L^1 \quad \text{and} \quad \eta^{A,A}: id_{F(A \times A)} \Rightarrow R^{A,A} \circ L^{A,A}$$

are isomorphisms

Corollary (Cigoli, Mantovani, M., 2023)

Let $\mathbf{B}$ be a category with finite products such that $\mathbf{Ab}(\mathbf{B}) \cong \mathbf{B}$, and $F: \mathbf{B} \rightarrow \underline{\mathbf{Cat}}$ be a lax cartesian monoidal pseudofunctor.

The two factorizations in the diagram imply each other.



OPEXT is a groupal pseudofunctor

Example (Extensions of groups with abelian kernel)

$$OPEXT: \mathbf{Mod}(C) \rightarrow \underline{\mathbf{Gpd}} \quad B \mapsto \left\{ \begin{array}{c} 0 \rightarrow B \begin{array}{c} \nearrow E \\ \searrow E' \end{array} \begin{array}{c} E \\ \vdots \\ E' \end{array} \nearrow C \rightarrow 0 \end{array} \right\}$$

gives the 2-groups $\mathbf{H}^2(C, B) := OPEXT(B)$, with $\pi_0(\mathbf{H}^2(C, B)) \cong H^2(C, B)$.

$$\begin{array}{ccc} B & B & B \\ \downarrow & \downarrow & \downarrow \\ E_1 & \otimes E_2 & = E_1 \times_C^B E_2 \\ \downarrow & \downarrow & \downarrow \\ C & C & C \end{array} \quad \text{where}$$

$$\begin{array}{ccccc} B \times B & \xlongequal{\quad} & B \times B & \xrightarrow{+} & B \\ \downarrow & & \downarrow & \text{p.f.} & \downarrow \\ E_1 \times E_2 & \xleftarrow{\quad} & E_1 \times_C E_2 & \xrightarrow{\quad} & E_1 \times_C^B E_2 \\ \downarrow & \text{p.b.} & \downarrow & & \downarrow \\ C \times C & \xleftarrow{\Delta_C} & C & \xlongequal{\quad} & C \end{array}$$

Is this the end? [of this talk]

I was to tell the story of "group preserving functors", shifted one dimension up.

For $\mathbf{B} \cong \mathbf{Ab}(\mathbf{B})$, we established that a pseudofunctor $F: \mathbf{B} \rightarrow \mathbf{Cat}$

- either lifts to groupoids, equivalently to symmetric 2-groups: F is **groupal**
- or does not lift to groupoids, i.e. F **not groupal**

So, if F does not lift, what can we do with it?

To answer this question we'd better switch to (op)fibrations.

Fibrational viewpoint

There is a canonical way to translate a pseudofunctors into (op)fibrations: the Grothendieck-Bénabou construction.

Good news! Such translation agrees with the monoidal structures. It is the *monoidal Grothendieck construction* in [Moeller Vasilakopoulou 2020].

$$\begin{array}{ccc} \text{psMon}(\underline{\text{psFun}}) & \xrightleftharpoons{\quad} & \text{psMon}(\underline{\text{opFib}}) \\ \downarrow & \int & \downarrow \\ \underline{\text{psFun}} & \xrightleftharpoons{\quad} & \underline{\text{opFib}} \end{array}$$

Theorem (Cigoli, Mantovani, M., 2023)

Let $\mathbf{B}$ be a cat. with fin. prod., and $F: \mathbf{B} \rightarrow \underline{\mathbf{Cat}}$ be a pseudofunctor. TFAE:

- F is a **cartesian object** in $\underline{\text{psFun}}(\mathbf{B}, \underline{\mathbf{Cat}})$, i.e. $L \dashv R$ and $L^1 \dashv R^1$
- F is **lax cartesian monoidal** $(F, R, R^1): (\mathbf{B}, \times, I) \longrightarrow (\underline{\mathbf{Cat}}, \times, I)$ with $R^{A,B}(X, Y) = X \times Y$ and $R^1(\star) = 1$ in the total category $\mathbf{X} = \int_{\mathbf{B}} F$
- The category $\mathbf{X}$ has finite products, and the opfibration $P: \mathbf{X} \rightarrow \mathbf{B}$ is **cartesian monoidal** (i.e. P strictly preserves finite products and cocartesian maps are product-stable [Shulman 2008])

Theorem (Cigoli, Mantovani, M., 2023 - enhanced version)

Let $F: \mathbf{B} \rightarrow \mathbf{Cat}$ a lax cartesian monoidal pseudofunctor. TFAE:

- F is groupal
- $\forall A$ in $\mathbf{B}$ supporting a group structure, the maps η^1 and $\eta^{A,A}$ are iso
- $\forall A$ in $\mathbf{B}$ supporting a group structure, $\forall X$ over A , the maps $\Delta_X: X \rightarrow X \times X$ and $\tau_X: X \rightarrow 1$ are cocartesian w.r.t. $\int_{\mathbf{B}} F \rightarrow \mathbf{B}$

Turning a Cat-pseudofunctor into a Gpd-pseudofunctor amounts to turning an opfibration into an opfibration with groupoidal fibres.

Theorem (Cigoli, Mantovani, M.)

Let $\mathbf{X}$ and $\mathbf{B}$ with fin. prod. and $P: \mathbf{X} \rightarrow \mathbf{B}$ an opfibration strictly preserving them. If Q presents $\bar{\mathbf{X}}$ as the category of fractions of $\mathbf{X}$ w.r.t. P -vertical maps, then $\bar{P}: \bar{\mathbf{X}} \rightarrow \mathbf{B}$ is a cartesian monoidal opfibration with groupoidal fibres:

$$\begin{array}{ccccc} & & P & & \\ & \curvearrowright & & \curvearrowleft & \\ \mathbf{X} & \xrightarrow{Q} & \bar{\mathbf{X}} & \xrightarrow{\bar{P}} & \mathbf{B} \end{array}$$

Lemma (Kelly Lack Walters, 1993)

Let $\mathbf{X}$ be a category with finite products and Σ a class of maps of $\mathbf{X}$ containing identities and closed under products. Then also the category of fractions $\overline{\mathbf{X}} = \mathbf{X}[\Sigma^{-1}]$ has finite products, and $Q: \mathbf{X} \rightarrow \overline{\mathbf{X}}$ preserves finite products.

Proposition (Cigoli, Mantovani, M.)

Let $\mathbf{X}$ and $\mathbf{B}$ be categories with finite products, and $P: \mathbf{X} \rightarrow \mathbf{B}$ an opfibration strictly preserving them. Consider the factorization

$$\mathbf{X} \xrightarrow[\quad Q \quad]{\quad P \quad} \overline{\mathbf{X}} \xrightarrow[\quad \overline{P} \quad]{} \mathbf{B}$$

through the category of fractions of $\mathbf{X}$ w.r.t. the class of P -vertical morphisms. Then $\overline{P}: \overline{\mathbf{X}} \rightarrow \mathbf{B}$ is a cartesian monoidal opfibration with **groupoidal fibres**. If moreover $\mathbf{B}$ is additive, each fibre of $\overline{P}$ is endowed with a **symmetric 2-group structure**, and change-of-base functors are symmetric monoidal.

$XEXT$ is not groupal

Example (Crossed extensions of groups)

$$XEXT: \mathbf{Mod}(C) \rightarrow \mathbf{Cat} \quad B \mapsto \left\{ \begin{array}{c} 0 \rightarrow B \begin{array}{ccccc} & E_1 & \xrightarrow{\partial} & E_0 & \\ & \uparrow & & \downarrow & \\ & | & & | & \\ & \downarrow & & \downarrow & \\ & E'_1 & \xrightarrow{\partial'} & E'_0 & \\ & \uparrow & & \uparrow & \\ & | & & | & \\ & \downarrow & & \downarrow & \\ & E'_1 & \xrightarrow{\partial'} & E'_0 & \\ & \uparrow & & \uparrow & \\ & | & & | & \\ & \downarrow & & \downarrow & \\ & E'_1 & \xrightarrow{\partial'} & E'_0 & \end{array} \\ & \end{array} \right\}$$

gives just a monoidal category $XEXT(B)$, with $\pi_0(XEXT(B)) \cong H^3(C, B)$.

If we factor the opfibration P associated with the pseudofunctor $XEXT$, we obtain a new opfibration $\overline{P}$ fibred in groupoids.

$$\begin{array}{ccccc} & & P & & \\ & \nearrow & & \searrow & \\ XEXT(C) & \xrightarrow{Q} & \overline{XEXT}(C) & \xrightarrow{\overline{P}} & \mathbf{Mod}(C) \end{array}$$

and all its fibres are canonically endowed with a symmetric 2-group structure.

The third cohomology 2-group $\mathbf{H}^3(C, B)$

Example (Crossed extensions of groups and butterflies)

$$\overline{XEXT}: \mathbf{Mod}(C) \rightarrow \mathbf{Gpd} \quad B \mapsto \left\{ \begin{array}{c} \begin{array}{ccccc} & E_1 & \xrightarrow{\partial} & E_0 & \\ & \nearrow & \searrow & \searrow & \\ 0 \rightarrow B & & F & & C \rightarrow 0 \\ & \searrow & \nearrow & \nearrow & \\ & E'_1 & \xrightarrow{\partial'} & E'_0 & \end{array} \end{array} \right\}$$

gives the 2-groups $\mathbf{H}^3(C, B)$, with $\otimes =$ butterfly composition.

A butterfly $F: \partial \rightarrow \partial'$ between two crossed modules is a commutative diagram

$$\begin{array}{ccc} E_2 & \xrightarrow{\partial} & E_1 \\ & \searrow \kappa & \nearrow \delta \\ & F & \\ & \nearrow \iota & \searrow \gamma \\ E'_2 & \xrightarrow{\partial'} & E'_1 \end{array}$$

such that:

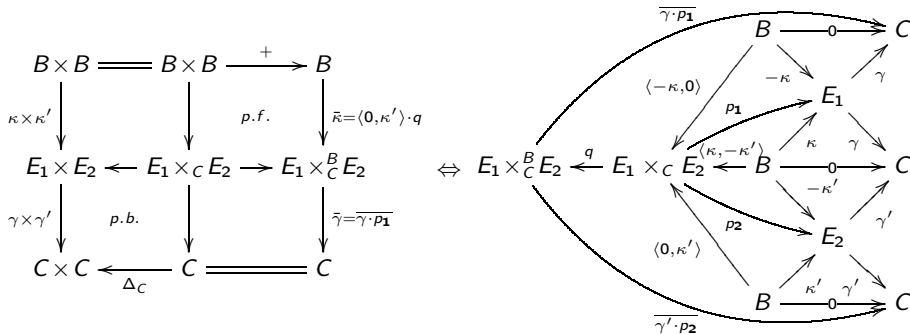
1. $\gamma \cdot \kappa = 0$
2. (δ, ι) is short exact
3. compatibility of actions

Proposition (Cigoli, Mantovani, M.)

For a C -module B , $\pi_0(\mathbf{H}^3(C, B)) \cong H^3(C, B)$ and $\pi_1(\mathbf{H}^3(C, B)) \cong H^2(C, B)$.

$Q_B: XEXT(C, B) \rightarrow \overline{XEXT}(C, B)$ is constant on objects and strict monoidal:
 $\pi_0(\mathbf{H}^3(C, B)) = \pi_0(XEXT(C, B)) \cong H^3(C, B)$.

$\pi_1(\mathbf{H}^3(C, B))$ is the group of loops of the identity object $B = B \xrightarrow{0} C = C$, and its auto-butterflies correspond to the group of extensions of C by B .



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THANK YOU FOR YOUR ATTENTION!