107th Peripatetic Seminar on Sheaves and Logic

April 1-2, 2023 *Athens, Greece*

Groupal pseudofunctors and cohomology 2-groups

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Intro

Fact

Every **Set**-*valued functor* F*, from an abelian category* **B***, that preserves finite products, factors through the category* **Ab** *of abelian groups.*

$$\begin{tikzcd} & \mathbf{Ab} \arrow[d, "U"] \\ \mathbf{B} \arrow[ur] \arrow[r, "F"'] & \mathbf{Set} \end{tikzcd}$$

Example: [Yoneda, 1960] defines the cohomology functors

$$Ext^n(A,-)\colon \mathbf{B} \longrightarrow \mathbf{Set}$$

$$B \mapsto \; Ext^n(A,B) = \{[0 \to B \to E_n \to \cdots \to E_1 \to A \to 0]_\sim\}$$

where $\sim$ is the **connectedness** relation.

What if we do not take the quotient on the connectedness relation?
We obtain a pseudofunctor $EXT^n(A,-)\colon \mathbf{B} \longrightarrow \mathbf{Cat}$.

Q: under what conditions on this pseudofunctor does **B** induce an additional structure on each category $EXT^n(A,B)$?

$$\begin{tikzcd} & \boxed{???} \arrow[d, "U"] \\ \mathbf{B} \arrow[ur] \arrow[r, "EXT^n(A,-)"'] & \mathbf{Cat} \end{tikzcd}$$

Intro

Fact

Every **Set***-valued functor F, from an abelian category* **B***, that preserves finite products factors through the category* **Ab** *of abelian groups.*

$$\begin{tikzcd} & \mathbf{Ab} \arrow[d, "U"] \\ \mathbf{B} \arrow[ur] \arrow[r, "F"'] & \mathbf{Set} \end{tikzcd}$$

This fact is a combination of two elementary observations:

(i) for **B** abelian, the forgetful functor $U\colon \mathbf{Ab}(\mathbf{B}) \to \mathbf{B}$ is an isomorphism

(ii) a **product preserving functor** takes (abelian) groups to (abelian) groups

the latter being a special instance of

(iii) a **monoidal functor** takes monoids to monoids

The aim of my talk is to explain how this whole story can be told in dimension 2

Since (i) will not be affected by our dimensional shift, we start from a well-known 2-categorical version of (iii), and then that of (ii).

First step: 2-categorical version of (iii)

Fact (Day Street 1997)

lax (2-)monoidal pseudofunctors take pseudomonoids to pseudomonoids:

$$\boxed{M \otimes M \xrightarrow{*} M} \mapsto \boxed{F(M) \otimes F(M) \xrightarrow{R^{M,M}} F(M \otimes M) \xrightarrow{F(*)} F(M)}$$

where a **pseudomonoid** *in a monoidal* 2*-category is a* **monoid object** $(M, *, e_M)$*, s.t. monoid axioms hold up to coherent isomorphis.*

If **B** has finite products, one can consider a pseudofunctor $F\colon \mathbf{B} \to \underline{\mathbf{Cat}}$, lax monoidal w.r.t the cartesian structures. Then F lifts:

$$\begin{tikzcd}
\mathbf{psMon(B)} = \mathbf{Mon(B)} \arrow[r, dashed] \arrow[d, "U"'] & \mathbf{psMon}(\underline{\mathbf{Cat}}) = \mathbf{MonCat} \arrow[d, "U"] \\
\mathbf{B} \arrow[r, "F"'] & \underline{\mathbf{Cat}}
\end{tikzcd}$$

First step: 2-categorical version of (iii)

If $\mathbf{B}$ is semiadditive ($\times = +$), the forgetful functor $U\colon \mathbf{CMon}(\mathbf{B}) \to \mathbf{B}$ is an iso, and F factors through the 2-category of symmetric monoidal categories:

$$\begin{tikzcd}
 & \underline{\mathbf{SymMC}} \arrow[d, "U"] \\
\mathbf{B} \arrow[ur] \arrow[r, "F"'] & \underline{\mathbf{Cat}}
\end{tikzcd}$$

Example

Consider the lax monoidal pseudofunctor

$$\mathrm{Sub}(-)\colon (\mathbf{Ab}, \oplus, 0) \to (\underline{\mathbf{Cat}}, \times, \mathbf{I})$$

that assigns to every abelian group A the poset $\mathrm{Sub}(A)$ of its subgroups.

Every A induces a symmetric monoidal structure on $\mathrm{Sub}(A)$, with tensor product given by the join of subgroups.

Second step: pseudogroups

Now we look for a result *a lá* Day-Street, which would state if/when a monoidal pseudofunctor $F\colon \mathbf{B} \to \underline{\mathbf{Cat}}$ takes internal groups to...

to what?

Definition (Baez Lauda, 2004)

A **pseudogroup** (*weak 2-group* in [*loc.cit.*]) in a 2-category $\underline{\mathbf{K}}$ with fin. prod. is a pseudomonoid $(G, \otimes, I)$ endowed with a 1-cell $(-)^*\colon G \to G$ and iso 2-cells with components $X \otimes X^* \to I$, $I \to X^* \otimes X$ satisfying coherence conditions.

In $\underline{\mathbf{Cat}}$, they are monoidal groupoids s.t. every object is $\otimes$-pseudoinvertible.

Why monoidal **groupoids***, and not just monoidal* **categories***?*

Fact: All the 1-cells of a pseudogroup $\mathbf{G}$ in $\underline{\mathbf{Cat}}$ are invertible w.r.t. composition, i.e. $\mathbf{G}$ is a groupoid. [Baez Lauda, 2004]

Reason: want $(-)^*\colon \mathbf{G} \to \mathbf{G}$ to be internal, i.e. a *covariant* functor. Requiring just pseudoinvertible objects produces a contravariant functor.

Eample: preordered groups.

Third step: groupal pseudofunctors

Recall that an object C of a bicategory with products is a **cartesian object** if the 1-cells $\Delta\colon C \to C \times C$ and $C \to 1$ have right adjoint.

Let $\underline{\mathbf{psFun}}$ be the 2-category (with pointwise defined fin. prod.) of pseudo-functors $F\colon \mathbf{B} \to \underline{\mathbf{Cat}}$, pseudocommutative triangles over $\underline{\mathbf{Cat}}$, and their morphisms.

Definition

A cartesian object in $\underline{\mathbf{psFun}}$ is called **lax cartesian monoidal pseudofunctor**.

In this the case, $\mathbf{B}$ has finite products, and F is lax monoidal

$$(F, R, R^1, \dots)\colon (\mathbf{B}, \times, 1) \to (\underline{\mathbf{Cat}}, \times, \mathbf{I})$$

where the components of the structural pseudonatural transformations are:

$$R^{A,B}\colon F(A) \times F(B) \to F(A \times B) \quad \text{and} \quad R^1\colon \mathbf{I} \to F(1)$$

and we won't mention the coherent modifications.

We'll see that $R^{A,B}(X, Y)$ and $R^1(\star)$ can be interpreted as binary and 0-ary products.

Third step: groupal pseudofunctors

Definition

A lax cartesian monoidal pseudofunctor $F\colon(\mathbf{B},\times,I)\to(\underline{\mathbf{Cat}},\times,\mathbf{I})$ is **groupal** if it lifts to a pseudofunctor $\hat{F}$:

$$\begin{tikzcd} \mathbf{Gp}(\mathbf{B}) \arrow[r, "\hat{F}"] \arrow[d, "U"'] & \underline{\mathbf{2Gp}} \arrow[d, "U"] \\ \mathbf{B} \arrow[r, "F"'] & \underline{\mathbf{Cat}} \end{tikzcd}$$

Fact

In this case, F also lifts to abelian groups:

$$\begin{tikzcd} \mathbf{Ab}(\mathbf{B}) \arrow[r, "\hat{\hat{F}}"] \arrow[d, "U"'] & \underline{\mathbf{Sym2Gp}} \arrow[d, "U"] \\ \mathbf{B} \arrow[r, "F"'] & \underline{\mathbf{Cat}} \end{tikzcd}$$

The lax cartesian monoidal pseudofunctor of subobjects:

$$\mathrm{Sub}(-)\colon(\mathbf{Ab},\oplus,0)\to(\underline{\mathbf{Cat}},\times,\mathbf{I})$$

shows that a result *a lá* Day-Street **does not hold** in general.
Counterexample: For $1\neq A\in\mathbf{Ab}$, the preorder $\mathrm{Sub}(A)$ is not a groupoid,
$\Rightarrow$ $\mathrm{Sub}(A)$ cannot support any 2-group structure.

AIM: characterize groupal pseudofunctors.

Proposition (Cigoli, Mantovani, M., 2023)

Let **B** *be a category with fin. prod., and* $F\colon \mathbf{B} \to \underline{\mathbf{Cat}}$ *be a pseudofunctor. Then* F *is canonically endowed with an* **oplax** *symmetric monoidal structure*

$$(F, L, L^1, \dots)\colon (\mathbf{B}, \times, I) \longrightarrow (\underline{\mathbf{Cat}}, \times, \mathbf{I})$$

$$\begin{tikzcd} \mathbf{B}\times\mathbf{B} \arrow[r, "\times"] \arrow[d, "F\times F"'] \arrow[dr, phantom, "\swArrow L"] & \mathbf{B} \arrow[d, "F"] \\ \underline{\mathbf{Cat}}\times\underline{\mathbf{Cat}} \arrow[r, "\times"'] & \underline{\mathbf{Cat}} \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathbf{I}} \arrow[r, "I"] \arrow[dr, "\mathbf{I}"'] & \mathbf{B} \arrow[d, "F"] \arrow[dl, phantom, "\swArrow L^1"] \\ & \underline{\mathbf{Cat}} \end{tikzcd}$$

$$L^{A,B}\colon F(A\times B) \to F(A)\times F(B)$$
$$Z \mapsto (F(\pi_1)(Z), F(\pi_2)(Z))$$

$$L^1\colon F(I) \to \mathbf{I}$$
$$J \mapsto *$$

Lemma (Cigoli, Mantovani, M.)

Let **B** *be a cat. with fin. prod., and* $F\colon \mathbf{B} \to \underline{\mathbf{Cat}}$ *be a pseudofunctor. TFAE:*

- F *is* **lax cartesian monoidal**, *i.e.* F *is a* **cartesian object** *in* $\underline{\textbf{psFun}}$
- L *and* L^1 *have a right adjoints* R *and* R^1

Theorem (Cigoli, Mantovani, M., 2023)

Let $\mathbf{B}$ *be a category with finite products, and* $F\colon \mathbf{B} \to \underline{\mathbf{Cat}}$ *a lax cartesian monoidal psudofunctor. TFAE:*

- F *is* **groupal**
- *For every* A *in* $\mathbf{B}$ *supporting an internal group structure, the maps*

$$\eta^1\colon id_{F(I)} \Rightarrow R^1 \circ L^1 \quad \text{and} \quad \eta^{A,A}\colon id_{F(A\times A)}) \Rightarrow R^{A,A} \circ L^{A,A}$$

  are isomorphisms

Corollary (Cigoli, Mantovani, M., 2023)

Let $\mathbf{B}$ *be a category with finite products such that* $\mathbf{Ab}(\mathbf{B}) \cong \mathbf{B}$*, and* $F\colon \mathbf{B} \to \underline{\mathbf{Cat}}$ *be a lax cartesian monoidal pseudofunctor.*

The two factorizations in the diagram imply each other.

$$\begin{tikzcd}
 & \underline{\mathbf{Sym2Gp}} \arrow[d, "U"] \\
 & \underline{\mathbf{Gpd}} \arrow[d, "U"] \\
\mathbf{B} \arrow[uur, dashed] \arrow[ur, dashed] \arrow[r, "F"'] & \underline{\mathbf{Cat}}
\end{tikzcd}$$

OPEXT is a groupal pseudofunctor

Example (Extensions of groups with abelian kernel)

$$OPEXT : \mathbf{Mod}(C) \to \underline{\mathbf{Gpd}} \qquad B \ \mapsto \left\{ \begin{tikzcd} & E \arrow[dd, dashed] \arrow[dr] & & \\ 0 \arrow[r] & B \arrow[ur] \arrow[dr] & & C \arrow[r] & 0 \\ & E' \arrow[ur] & & \end{tikzcd} \right\}$$

gives the 2-groups $\mathbf{H}^2(C,B) := OPEXT(B)$, with $\pi_0\left(\mathbf{H}^2(C,B)\right) \cong H^2(C,B)$.

$$\begin{tikzcd} B \arrow[d] \\ E_1 \arrow[d] \\ C \end{tikzcd} \otimes \begin{tikzcd} B \arrow[d] \\ E_2 \arrow[d] \\ C \end{tikzcd} = \begin{tikzcd} B \arrow[d] \\ E_1 \times_C^B E_2 \arrow[d] \\ C \end{tikzcd} \quad \text{where} \quad \begin{tikzcd} B \times B \arrow[r, equal] \arrow[d] & B \times B \arrow[r, "+"] \arrow[d] \arrow[dr, phantom, "p.f."] & B \arrow[d] \\ E_1 \times E_2 \arrow[d] \arrow[dr, phantom, "p.b."] & E_1 \times_C E_2 \arrow[l] \arrow[r] \arrow[d] & E_1 \times_C^B E_2 \arrow[d] \\ C \times C & C \arrow[l, "\Delta_C"'] \arrow[r, equal] & C \end{tikzcd}$$

Is this the end? [of this talk]

I was to tell the story of "group preserving functors", shifted one dimesion up.

For $\mathbf{B} \cong \mathbf{Ab}(\mathbf{B})$, we established that a pseudofunctor $F\colon \mathbf{B} \to \underline{\mathbf{Cat}}$

- either lifts to groupoids, equivalently to symmetric 2-groups: F is **groupal**
- or does not lift to groupoids, i.e. F **not groupal**

So, if F does not lift, what can we do with it?

To answer this question we'd better switch to (op)fibrations.

Fibrational viewpoint

There is a canonical way to translate a pseudofunctors into (op)fibrations: the Grothendieck-Bénabou construction.

Good news! Such translation agrees with the monoidal structures. It is the *monoidal Grothedieck construction* in [Moeller Vasilakopoulou 2020].

$$\begin{tikzcd}
\mathbf{psMon}(\underline{\mathbf{psFun}}) \arrow[r, shift left] \arrow[d] & \mathbf{psMon}(\underline{\mathbf{opFib}}) \arrow[l, shift left] \arrow[d] \\
\underline{\mathbf{psFun}} \arrow[r, shift left, "\int"] & \underline{\mathbf{opFib}} \arrow[l, shift left]
\end{tikzcd}$$

Theorem (Cigoli, Mantovani, M., 2023)

Let $\mathbf{B}$ *be a cat. with fin. prod., and* $F\colon \mathbf{B} \to \underline{\mathbf{Cat}}$ *be a pseudofunctor. TFAE:*

- F *is a* **cartesian object** *in* $\underline{\mathbf{psFun}}(\mathbf{B}, \underline{\mathbf{Cat}})$, *i.e.* $L \dashv R$ *and* $L^1 \dashv R^1$
- F *is* **lax cartesian monoidal** $(F, R, R^1)\colon (\mathbf{B}, \times, I) \longrightarrow (\underline{\mathbf{Cat}}, \times, \mathbf{I})$ *with* $R^{A,B}(X, Y) = X \times Y$ *and* $R^1(\star) = 1$ *in the total category* $\mathbf{X} = \int_{\mathbf{B}} F$
- *The category* $\mathbf{X}$ *has finite products, and the opfibration* $P\colon \mathbf{X} \to \mathbf{B}$ *is* **cartesian monoidal** *(i.e.* P *strictly preserves finite products and cocartesian maps are product-stable [Shulman 2008])*

Theorem (Cigoli, Mantovani, M., 2023 - enhanced version)

Let $F\colon \mathbf{B} \to \underline{\mathbf{Cat}}$ a lax cartesian monoidal psudofunctor. TFAE:

- *F is* **groupal**
- *$\forall A$ in $\mathbf{B}$ supporting a group structure, the maps η^1 and $\eta^{A,A}$ are iso*
- *$\forall A$ in $\mathbf{B}$ supporting a group structure, $\forall X$ over A, the maps $\Delta_X\colon X \to X \times X$ and $\tau_X\colon X \to 1$ are cocartesian w.r.t. $\int_{\mathbf{B}} F \to \mathbf{B}$*

Turning a $\underline{\mathbf{Cat}}$-pseudofuctor into a $\underline{\mathbf{Gpd}}$-pseudofuctor amounts to turning an opfibration into an opfibration with groupoidal fibres.

Theorem (Cigoli, Mantovani, M.)

Let $\mathbf{X}$ and $\mathbf{B}$ with fin. prod. and $P\colon \mathbf{X} \to \mathbf{B}$ an opfibration strictly preserving them. If Q presents $\overline{\mathbf{X}}$ as the category of fractions of $\mathbf{X}$ w.r.t. P-vertical maps, then $\overline{P}\colon \overline{\mathbf{X}} \to \mathbf{B}$ is a cartesian monoidal opfibration with groupoidal fibres:

$$\begin{tikzcd} \mathbf{X} \arrow[r, "Q"'] \arrow[rr, bend left, "P"] & \overline{\mathbf{X}} \arrow[r, "\overline{P}"'] & \mathbf{B} \end{tikzcd}$$

Lemma (Kelly Lack Walters, 1993)

Let $\mathbf{X}$ *be a category with finite products and* Σ *a class of maps of* $\mathbf{X}$ *containing identities and closed under products. Then also the category of fractions* $\overline{\mathbf{X}} = \mathbf{X}[\Sigma^{-1}]$ *has finite products, and* $Q\colon \mathbf{X} \to \overline{\mathbf{X}}$ *preserves finite products.*

Proposition (Cigoli, Mantovani, M.)

Let $\mathbf{X}$ *and* $\mathbf{B}$ *be categories with finite products, and* $P\colon \mathbf{X} \to \mathbf{B}$ *an opfibration strictly preserving them. Consider the factorization*

$$\begin{tikzcd} \mathbf{X} \arrow[r, "Q"'] \arrow[rr, bend left, "P"] & \overline{\mathbf{X}} \arrow[r, "\overline{P}"'] & \mathbf{B} \end{tikzcd}$$

through the category of fractions of $\mathbf{X}$ *w.r.t. the class of* P*-vertical morphisms. Then* $\overline{P}\colon \overline{\mathbf{X}} \to \mathbf{B}$ *is a cartesian monoidal opfibration with* **groupoidal fibres***.*
If moreover $\mathbf{B}$ *is additive, each fibre of* $\overline{P}$ *is endowed with a* **symmetric 2-group structure***, and change-of-base functors are symmetric monoidal.*

$XEXT$ is not groupal

Example (Crossed extensions of groups)

$$XEXT\colon \mathbf{Mod}(C) \to \underline{\mathbf{Cat}} \qquad B \mapsto \left\{ \begin{tikzcd} & E_1 \arrow[r, "\partial"] \arrow[d, dashed] & E_0 \arrow[dr] \arrow[d, dashed] & \\ 0 \arrow[r] & B \arrow[ur] \arrow[dr] & & C \arrow[r] & 0 \\ & E_1' \arrow[r, "\partial'"'] & E_0' \arrow[ur] & \end{tikzcd} \right\}$$

gives just a monoidal category $XEXT(B)$, with $\pi_0\,(XEXT(B)) \cong H^3(C, B)$.

If we factor the opfibration P associated with the pseudofunctor $XEXT$, we obtain a new opfibration $\overline{P}$ fibred in groupoids.

$$\begin{tikzcd} \mathbf{XEXT}(C) \arrow[r, "Q"'] \arrow[rr, bend left, "P"] & \overline{\mathbf{XEXT}}(C) \arrow[r, "\overline{P}"'] & \mathbf{Mod}(C) \end{tikzcd}$$

and all its fibres are canonically endowed with a symmetric 2-group structure.

The third cohomology 2-group $\mathbf{H}^3(C, B)$

Example (Crossed extensions of groups and butterflies)

$$\overline{XEXT} \colon \mathbf{Mod}(C) \to \underline{\mathbf{Gpd}} \qquad B \mapsto \left\{ \begin{tikzcd} & & E_1 \arrow[rr, "\partial"] \arrow[dr] & & E_0 \arrow[dr] & & \\ 0 \arrow[r] & B \arrow[ur] \arrow[dr] & & F \arrow[ur] \arrow[dr] & & C \arrow[r] & 0 \\ & & E_1' \arrow[rr, "\partial'"'] \arrow[ur] & & E_0' \arrow[ur] & & \end{tikzcd} \right\}$$

gives the 2-groups $\mathbf{H}^3(C, B)$, with $\otimes =$ butterfly composition.

A butterfly $F \colon \partial \to \partial'$ between two crossed modules is a commutative diagram

$$\begin{tikzcd} E_2 \arrow[rr, "\partial"] \arrow[dr, "\kappa"'] & & E_1 \\ & F \arrow[ur, "\delta"'] \arrow[dr, "\gamma"] & \\ E_2' \arrow[rr, "\partial'"'] \arrow[ur, "\iota"] & & E_1' \end{tikzcd}$$

such that:

1. $\gamma \cdot \kappa = 0$
2. (δ, ι) is short exact
3. compatibility of actions

Proposition (Cigoli, Mantovani, M.)

For a C-module B, $\pi_0\left(\mathbf{H}^3(C,B)\right) \cong H^3(C,B)$ and $\pi_1\left(\mathbf{H}^3(C,B)\right) \cong H^2(C,B)$.

$Q_B \colon XEXT(C,B) \to \overline{XEXT}(C,B)$ is constant on objects and strict monoidal:
$\pi_0\left(\mathbf{H}^3(C,B)\right) = \pi_0\left(XEXT(C,B)\right) \cong H^3(C,B)$.

$\pi_1\left(\mathbf{H}^3(C,B)\right)$ is the group of loops of the identity object $B = B \xrightarrow{0} C = C$,
and its auto-butterflies correspond to the group of extensions of C by B.

$$
\begin{tikzcd}
B\times B \arrow[r, equal] \arrow[d, "\kappa\times\kappa'"'] & B\times B \arrow[r, "+"] \arrow[d] \arrow[dr, phantom, "p.f."] & B \arrow[d, "\bar\kappa=\langle 0,\kappa'\rangle\cdot q"] \\
E_1\times E_2 \arrow[d, "\gamma\times\gamma'"'] \arrow[dr, phantom, "p.b."] & E_1\times_C E_2 \arrow[l] \arrow[r] \arrow[d] & E_1\times^B_C E_2 \arrow[d, "\bar\gamma=\overline{\gamma\cdot p_1}"] \\
C\times C & C \arrow[l, "\Delta_C"] \arrow[r, equal] & C
\end{tikzcd}
\quad\Leftrightarrow\quad
\begin{tikzcd}
 & & B \arrow[rr, "0"] \arrow[dr, "-\kappa"] \arrow[ddl, "\langle -\kappa,0\rangle"'] & & C \\
 & & & E_1 \arrow[ur, "\gamma"'] \arrow[dr, "\gamma"] & \\
E_1\times^B_C E_2 \arrow[uurrrr, bend left, "\overline{\gamma\cdot p_1}"] \arrow[ddrrrr, bend right, "\overline{\gamma'\cdot p_2}"'] & E_1\times_C E_2 \arrow[l, "q"'] \arrow[urr, "p_1"] \arrow[drr, "p_2"'] & B \arrow[l, "\langle\kappa,-\kappa'\rangle"'] \arrow[ur, "\kappa"] \arrow[dr, "-\kappa'"'] \arrow[rr, "0"] & & C \\
 & & & E_2 \arrow[ur, "\gamma'"'] \arrow[dr, "\gamma'"] & \\
 & & B \arrow[uul, "\langle 0,\kappa'\rangle"] \arrow[ur, "\kappa'"] \arrow[rr, "0"] & & C
\end{tikzcd}
$$

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THANK YOU FOR YOUR ATTENTION!