Relative ideals, homological categories and non-classical logics

Serafina Lapenta[1], Giuseppe Metere[2], and Luca Spada[1]*

[1] Università degli Studi di Salerno – Via Giovanni Paolo II, 132 – 84084 Fisciano (SA), Italy.
{slapenta,lspada}@unisa.it

[2] Università degli Studi di Milano – Via Luigi Mangiagalli, 25 – 20133 Milano, Italy.
giuseppe.metere@unimi.it

0 Basic terminology in homological category theory

We briefly recall the following key concepts. We refer the interested reader to [2, 3] for a comprehensive account.

A category is *pointed* if it has a zero-object, that is, an object 0 that is both terminal and initial. In a pointed category, a *kernel* of a map $p: A \to B$ is the pullback of the initial map $0 \to B$ along p.

$$\begin{array}{ccc} \mathrm{Ker} & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle p} \\ 0 & \longrightarrow & B \end{array}$$

More generally, a *kernel pair* of p is the pullback of p with itself. An (internal) equivalence relation is called *effective* when it is the kernel pair of a morphism. A category B is *regular*, if it is finitely complete, has pullback-stable regular epimorphisms, and all effective equivalence relations admit coequalizers. A regular category is *Barr-exact* [1] when all equivalence relations are effective.

We will also need the concept of *protomodularity* whose technical definition goes beyond the scope of this abstract. Intuitively, a category is called *protomodular* if it possesses an intrinsic notion of *normal subobject* —in analogy with normal subgroups. The notion of protomodular (and semi-abelian) category encompasses a wide range of categories of interest to algebraists, including categories of groups, rings, Lie algebras, associative algebras, and cocommutative Hopf algebras, among others. Of course, all abelian categories are semi-abelian. We refer the reader to [4] for the original definition and to [5] for a more general account.

A category is called *homological* if it is pointed, regular and protomodular. Many of the standard results of classical homological algebra hold in homological categories.

Finally, a category is *semi-abelian* if it is homological, Barr-exact, and has finite coproducts.

1 Introduction

In the algebraic semantics of non-classical logics, often there are two constants in the language, one for absolute truth and one for absolute falsehood. As a consequence, the final object (=the trivial algebra) is different from the initial one (=the $\emptyset$-generated free algebra). This makes the categories at play non-pointed and thus apparently intractable with homological methods. We report on the article [9], which aims at showing that one may still connect the equivalent

*Presenting author

algebraic semantics of non-classical logics to homological categories. The idea is best explained by the following example from classical algebra.

The forgetful functor

$$U\colon \mathsf{CRing} \to \mathsf{CRng}\,,$$

from the protomodular category of commutative unital rings to the semi-abelian category of commutative rings has a left adjoint F that freely adds the multiplicative identity. In greater detail, if R is a commutative ring, then $F(R)$ has underlying abelian group given by the direct product $R \times \mathbb{Z}$, endowed with a multiplication defined by the formula

$$(r, n)(r', n') := (rn' + nr' + rr', nn') \tag{1}$$

and with multiplicative identity $(0, 1)$. It is easy to see that the ring R is contained in $F(R)$ as an ideal. By the arbitrary choice of R, this means that every commutative ring can be seen as an ideal of a suitable *unital* commutative ring. More precisely, one can prove that there is an equivalence of categories $\mathsf{CRing}/\mathbb{Z} \simeq \mathsf{CRng}$ which is defined by taking kernels in CRng of the "objects" in the slice category.

Now, it is well known that ideals of commutative unital rings make it easier to deal with congruences (and quotients) in CRing, but since they are not subobjects in CRing, it is not immediate to describe them categorically. On the other hand, ideals of commutative unital rings are subobjects in CRng. This makes it possible to exploit the categorical properties of the semi-abelian category CRng in order to study the, still protomodular but not pointed, category CRing.

2 U-ideals and the basic setting

We propose to set a study of these facts in a more general framework. As a first step, we introduce the notion of *U-ideal*. Let A be a pointed category with pullbacks and $U\colon \mathsf{B} \to \mathsf{A}$ be a faithful functor. Intuitively, a U-ideal is a kernel in A of a map that lives in B. A formal definition follows.

Definition 2.1. Let B be an object in B. A morphism $k\colon A \to U(B)$ in A is called *U-ideal* of B if there exists a morphism $f\colon B \to B'$ of B that makes the following diagram a pullback in A:

$$\begin{array}{ccc} A & \xrightarrow{\;k\;} & U(B) \\ \downarrow & & \downarrow{\scriptstyle U(f)} \\ 0 & \longrightarrow & U(B') \end{array}$$

Since kernels are monomorphisms, U-ideals can be seen as subobjects in a "larger" category. Prototypical examples of U-ideals come from the inclusion $U\colon \mathsf{Ring} \to \mathsf{Rng}$ of unital rings into rings: U-ideals in Ring are just the usual bilateral ideals of ring theory.

The example from ring theory suggests considering a more robust environment for dealing with the notion of U-ideal.

Definition 2.2. A *basic setting* for relative U-ideals is an adjunction

$$\mathsf{B} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{A}\,, \tag{2}$$

where the category A is homological and U is a conservative faithful functor.

We develop these ideas below. Our main results are the following. An equivalence, in the varietal settings, between relative U-ideals and Ursini's 0-ideals [7] (see Theorem 3.4). In Section 4 we describe a more general equivalence (Theorem 4.2) and then we apply it to the case of MV-algebras, showing an equivalence between Weisberg hoops and filters of MV-algebras (see Corollary 4.4). In order to apply our framework to MV-algebras we show that the category of Hoops is homological (see Theorem 4.3), thus showing that our setting applies to a number of non-classical logics.

3 U-ideals and varieties of algebras

Let us recall the characterization of protomodular varieties, as established by Bourn and Janelidze in [6].

Theorem 3.1. *A variety* V *of universal algebras is protomodular if and only if there is a natural number* n*,* 0*-ary terms* $e_1, \ldots, e_n$*, binary terms* $\alpha_1, \ldots, \alpha_n$*, and* $(n+1)$*-ary term* θ *such that:*

$$\begin{aligned} &\mathsf{V} \models \theta(\alpha_1(x,y), \ldots, \alpha_n(x,y), y) = x && \textit{and} \\ &\mathsf{V} \models \alpha_i(x,x) = e_i && \textit{for } i = 1, \ldots, n\,. \end{aligned} \tag{3}$$

Let us also recall the following definitions introduced by Ursini (see [7] and references therein).

Definition 3.2. Let V be a variety with a constant symbol 0 in its signature Σ_{V}. A Σ_V-term $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is called 0-*ideal term* in the variables $y_1, \ldots, y_n$ if

$$\mathsf{V} \models t(x_1, \ldots, x_m, 0, \ldots, 0) = 0\,.$$

For any algebra A in V, a subset $\emptyset \neq H \subseteq A$ is called 0-*ideal* if for every $a_1, \ldots, a_m \in A$, any $h_1, \ldots, h_n \in H$, and every 0-ideal term $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ in the variables $y_1, \ldots, y_n$, one has $t(a_1, \ldots, a_m, h_1, \ldots, h_n) \in H$.

It turns out that in any algebra in V the equivalence class of 0 under any given congruence is a 0-ideal. Vice versa, one calls 0-*ideal determined*, or just *ideal determined* (cf. [7, Definition 1.3]) a variety V where every 0-ideal is the equivalence class of 0 for a unique congruence relation.

A relevant subclass of the class of ideal determined varieties is the class of the so-called *classically ideal determined*. They are the varieties satisfying the conditions of Theorem 3.1, with all e_i's equal to a single constant $0 \in \Sigma_{\mathsf{V}}$.

In order to compare our notion of U-ideal and Ursini's notion of 0-ideal, we specialize the basic setting of Definition 2.2 to the varietal case.

Definition 3.3. A *basic setting for varieties* is given by a functor $U\colon \mathsf{B} \to \mathsf{A}$ between two varieties of algebras such that: 1. the axioms of B extend the axioms of A, possibly in a larger language; 2. the category A is homological (thus semi-abelian); 3. the functor U is the obvious forgetful functor.

Notice that, according to [2, Proposition 3.5.7 and Theorem 3.7.7], U is a faithful conservative right adjoint, thus satisfies the conditions of Definition 2.2. Obviously, if $U\colon \mathsf{B} \to \mathsf{A}$ is a basic setting for varieties, then A is classically ideal determined. The next result establishes a connection between the varietal notion of 0-ideal and the categorical notion of U-ideal.

Theorem 3.4. *Let* $U\colon \mathsf{B} \to \mathsf{A}$ *be a basic setting for varieties. A subset* H *of an algebra* B *of* B *is a* 0*-ideal of* B *if and only if* $H \subseteq U(B)$ *is a* U*-ideal of* B *with respect to* $U\colon \mathsf{B} \to \mathsf{A}$*.*

4 Categorical equivalence and non classical logics

Let us go back to the general situation and consider a basic setting as in (2). For every object A of A, the unit of the adjunction η gives a universal morphism $\eta_A \colon A \to UF(A)$. Thus, for any B in B the unique morphism $0 \colon A \to 0 \to U(B)$ factors through η_A as in the diagram below:

$$\begin{array}{ccc} F(A) & & A \xrightarrow{\ \eta_A\ } UF(A) \\ {\scriptstyle \exists! p_A}\downarrow & \text{such that} & {\scriptstyle 0}\searrow \quad \downarrow {\scriptstyle U(p_A)} \\ B & & U(B) \end{array} \tag{4}$$

Definition 4.1. We say that η_A is an *augmentation U-ideal*, or more simply an *augmentation ideal*, if it is the kernel of $U(p_A)$.

Theorem 4.2. *Suppose that in the basic setting* (2) *for every A in A, the component η_A is an augmentation ideal. Then, the kernel functor*

$$K \colon \mathsf{B}/B \to \mathsf{A}$$

defined on objects by letting $K(f) := Ker(U(f))$, is an equivalence of categories.

Finally, we apply Theorem 4.2 to the setting of MV-algebras. First, generalising a proof in [8] we provide terms as requested by Theorem 3.1, thus obtaining the following result.

Theorem 4.3. *The variety* Hoops *is semi-abelian.*

Consequently, our setting applies to certain categories of interest in algebraic logic, such as Wajsberg, Product, and Gödel hoops, with respect to MV, Product and Gödel algebras. In particular, an application of Theorem 4.2 gives the following.

Corollary 4.4. *The kernel functor $K \colon \mathsf{MVAlg}/\mathbf{2} \to \mathsf{WHoops}$ defined by sending an arbitrary homomorphism $f \colon A \to \mathbf{2}$ into its kernel $Ker(f)$, is an equivalence of categories.*

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