Algorithms and diagrammatic reasoning.

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Responsibility in the Knowledge Production and its Applications
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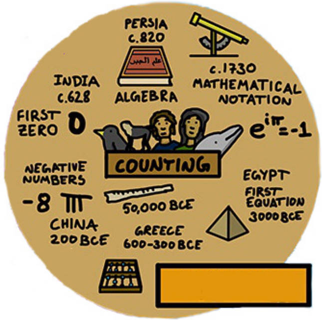
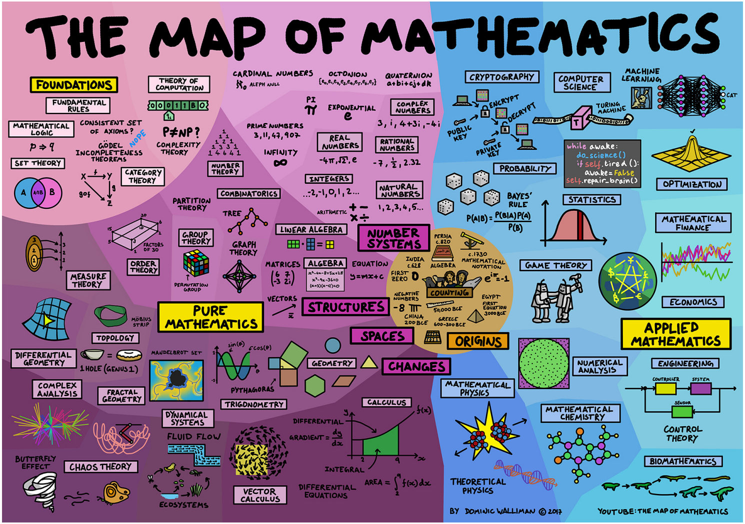
Introduction

Mathematics is often perceived as a static array of knowledge and skills



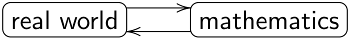
from "The map of mathematics", by Dominic Walliman, 2017.



However, just like other creations of mankind, mathematics evolves in a dialectic relationship with the human society and its issues.

Mathematics and the human development

Real world and mathematics interact along their timelines.



Two classical examples:

- maths for engineering: continuous mathematics, calculus...
- maths for computer science: discrete maths, algebra...

But also, geometry, statistics, game theory, control theory etc. for life sciences, social sciences etc.

In fact the chain

MATHS serves SCIENCE serves REAL WORLD

is a closed chain:

Real world is a primary source of inspiration for even the more abstract branches of mathematics.

Today maths for tomorrow science

> *"The majority of current mathematical approaches that have been applied to industries and societies are based solely on 19th-century mathematics. This means that the field of mathematics after the beginning of the 20th century is a broad, undeveloped frontier area, which no one has ever cultivated."*
>
> *The Coming Era of Mathematical Capitalism - How the Power of Mathematics Changes Our Future*
>
> *Report from Industry-Academia Round-Table Discussion at the Ministry of Economics Trade and Industry, Japan, 2019.*

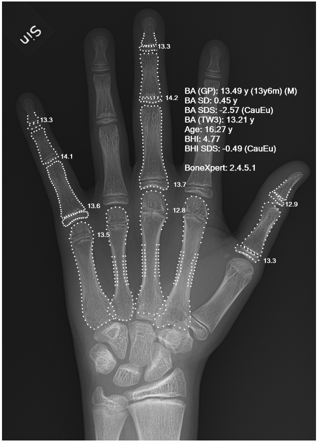
Here, the term *mathematical capitalism* has keywords: e-economy, virtual currencies, dematerialization, profiling, AI-boom etc.

But, is this the unique perspective to deal with, when we think of the development of mathematics in the near future?

What different keywords would we propose?

I would like to pick one from the title of THIS workshop...

Keyword: RESPONSABILITY



What system theory?

Systems theories are relevant places where **maths meets science** by providing models of the phenomena under investigation.

Physics exercises we faced at school were often dealt with by means of **closed** and **continuous** (dynamical) systems. The idea was to consider the experiment as taking place in an idealized isolated setting, where the time variable varies in a continuous way over the real numbers.

Discrete vs Continuous: *The rise of IT has spread the notion of discrete time, and consequently of discrete dynamical systems.*

Open vs Closed: *The systems we aim to describe require energy and resources, and they produce goods and. . . rubbish! Therefore they interact with the environment around them.*

What system theory?

These two dichotomies don't ask us for a choice, but for a **synthesis**:

- capable of facing the complexity of mixed systems where digital data and continuous data are just parts of a network of resources of different kinds;
- such that complex systems interact with the environment as well as their constituting subsystems interact with each other.

Moreover, such a synthesis should cope with

- uncertainty, chance, delay, human reactions, social interactions, etc.

Of course **nobody has answers or recipes** for such impressively deep and wide issues, but, as a mathematician I try to figure out how my community can contribute to the development of the mathematics of our parlous times.

Disclaimer

My professional interests in mathematics concern mainly the realm of abstract algebraic structures, with a conceptual approach to what is often considered a (boring) formal manipulation of symbols.

The main field of my research is **Category Theory**.

Category Theory was invented by Samuel Eilenberg and Saunders MacLane between the 40's and the 50's of 1900. It is a branch of mathematics that many mathematicians themselves consider rather abstract. So, why should category theory be helpful to our purposes?

(Not only) recently, a new trend emerged, that promotes a categorical approach to the study of complex interconnected mixed systems.

The rest of my talk is devoted to giving an account of it.

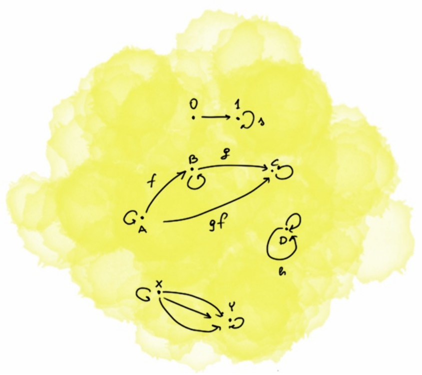
I am not claiming that this approach is THE right one for the forthcoming challenges of our times, however it MIGHT be of interest for who is in charge of knowledge production.

P.S.: I tried to be not-too-technical and not-too-naive, so I will make unhappy both the experts and the non-expert! Sorry for that!

Category Theory

A key idea in the categorical theoretic approach is that it is possible to study the properties of a mathematical object by observing its connections with all the other objects around it.

Its formalism is ...ehm, formal! Here, we provide a non-technical outline.



A category consists of two different sorts of entities:
objects and **morphisms**.

Morphisms can be composed.

The composition rules form the **structure** of a category.

Category Theory

Usually, the objects of a category are specified mathematical entities. They are represented by dots

$$\bullet \qquad \bullet$$

so that one is not allowed to investigate their properties by watching inside them.

On the other hand morphisms connect such dots in a suitable way.

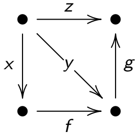
$$\bullet \longrightarrow \bullet$$

Morphisms relate objects that share the same $\mu o \rho \varphi \acute{\eta}$ (shape, form, type). So for instance, morphisms between algebraic entities preserve their algebraic structure, morphisms between geometric entities preserve (some specified) geometric shape, and so on.

Category Theory

Usually, calculations are performed in a line: start with a mathematical expression, then write the *equal* sign, then do some magic (=algebraic) manipulations and write the resulting modified expression. This process can be iterated, until one reaches the desired result.

Calculations in a category can be performed by using diagrams, e.g.



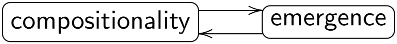
$$\text{reads: } \begin{cases} \text{if } f(x) = y \text{ and } g(y) = z \text{ then } gf(x) = z \\ \qquad\qquad \text{or} \\ \text{if } xf = y \text{ and } yg = z \text{ then } xfg = z \end{cases}$$

This adds more dimensions to calculations (in this case one more dimension, since the diagram is two-dimensional) and it represents a first naive idea of **diagrammatic reasoning**.

Category Theory

In fact, more structure can be defined on a category, than just the composition of morphisms. Such structured categories come with a built-in **formal setting for studying complex systems**, that allows us to investigate the ways, the patterns, the algebra of the relations among different entities that form the system itself.

Before giving some details of what is stated above, let us highlight two interacting features that categorical system theories take into account.



See *J. Hedges, Towards compositional game theory, PhD thesis, QMU London, 2016.*

Compositionality

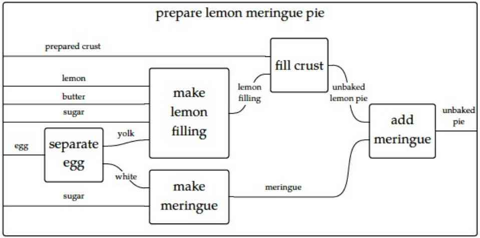
> *"A key factor [of CPS design] is compositionality, [...] the ability to infer the performance and quality of the entire system from its components".*
>
> *"Scientific and technical challenges to achieving compositionality include a lack of mathematical and system science foundations".*
>
> *J. Sztipanovits and S. Ying. Strategic r&d opportunities for 12st century Cyber-Physical Systems.*
>
> *Technical report, National Institute of Standards and Technology, USA, 2013.*

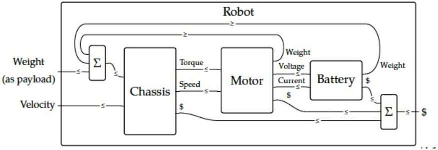
Well, is this just reductionism? Not precisely. Ask Wikipedia...

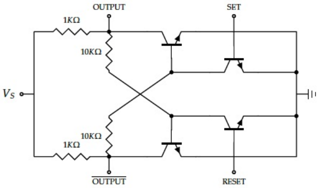
> *"In mathematics, semantics, and philosophy of language, the principle of compositionality is the principle that the meaning of a complex expression is determined by the meanings of its constituent expressions and the rules used to combine them."*

When exported to system theory, this principle would state the importance of the BOTH **constituents** of a system AND of the **interconnections** among them.

Compositionality: examples



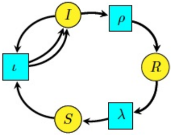


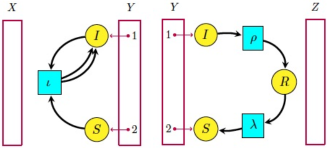


Three system diagrams, from: *B. Fong and D. Spivak, An Invitation to Applied Category Theory: Seven Sketches in Compositionality, Cambridge University Press, 2019.*

Compositionality: examples

$$S + I \xrightarrow{\iota} 2I \qquad\qquad I \xrightarrow{\rho} R \xrightarrow{\lambda} S$$





The SIRS model, from: *J. C. Baez and B. S. Pollard, A compositional framework for reaction networks, Reviews in Math. Ph. 29 (09), 2017.*

Compositionality and CT

Recall a **category** consists of a bunch of objects (dots) and morphisms (arrows). Take a morphism

$$\overset{A}{\bullet} \xrightarrow{f} \overset{B}{\bullet}$$

and suppose we want it to represent a process f.

This is all fine, provided we are to describe a process f with a single input A and a single output B. However, (sub)systems considered before have more involved interfaces, than just a single input and a single output.

For Category Theory to deal with compositional systems, we need

- more structure on the categories concerned.
- a change the way we draw objects and arrows of a category;

Monoidal Categories

We can provide a category with additional structures.

A **monoidal structure** on a category is a (unitary) multiplication law for both the objects and the morphisms of the category.
Given objects A and X we can form their product $A \otimes X$, given morphisms f, m we can also form their product $f \otimes m$, which is consistent with the product of the objects:

$$\left(\overset{A}{\bullet} \xrightarrow{f} \overset{B}{\bullet} \right) \otimes \left(\overset{X}{\bullet} \xrightarrow{m} \overset{Y}{\bullet} \right) \quad = \quad \overset{A \otimes X}{\bullet} \xrightarrow{f \otimes m} \overset{B \otimes Y}{\bullet}$$

Easy? Well. . . not so. I am hiding the details under the carpet!

Monoidal categories (aka **tensor categories**) are a well known ubiquitous gadget in mathematics. There are too many references to list.

A milestone: S. MacLane, Categories for the working mathematician, Springer, 1971.

A survey: Ross Street, Monoidal categories in, and linking, geometry and algebra, arXiv:1201.2991.

Monoidal Categories

Back to our problem of modeling processes with multiple interfaces, monoidal categories can accomodate this!

For instance if our process "*sys*" has

- two inputs I_1 and I_2
- three outputs O_1, O_2 and O_3,

we can consider the morphism

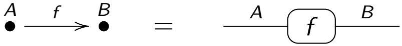
$$I_1 \otimes I_2 \xrightarrow{sys} O_1 \otimes O_2 \otimes O_3$$

Great!

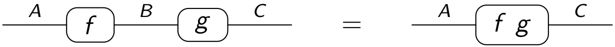
But now, the graphic language we are using is no longer adeguate!

String diagrams

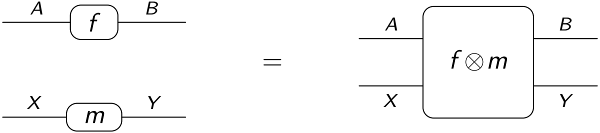
A solution comes from Poincaré duality. Just turn dots in lines and lines in dots, ehm, **big** dots!



Deals with composition of morphisms...



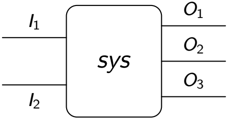
...and with ⊗ multiplication...



String diagrams

Our example process "*sys*" looks now more familiar:

$$I_1 \otimes I_2 \xrightarrow{sys} O_1 \otimes O_2 \otimes O_3 \qquad =$$

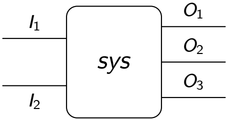


Of course, we gave **just an idea** of this graphic language.

- The real thing is necessarily more involved, but extremely expressive.
- It can not only describe the syntax of compositional systems, but it allows us to do calculations!

Relational vs functional approach to behaviour.

Consider the terminals I_1, I_2 and O_1, O_2, O_3 of the example:



If this is a part of a description of a system in which something is flowing from the I's to the O's, then it is ok to speak of *inputs* and *outputs*.

However, in many complex systems, there is not a specified direction of *flowing*, or, more simply, nothing actually flows!

Then, it is better to consider the terminals as **interfaces**, where the different subsystems get in touch and possibly exchange some sort information. This happens, e.g. for electric circuits, chemical reactions, ecosystems, etc.

A mathematical description of such interfaces is dealt with so-called **decorated cospans** in *B. Fong,* The Algebra of Open and Interconnected Systems. *PhD thesis, University of Oxford, 2016.*

Emergence

Do we live in a compositional world?
So far, we cherry-picked some instances of possibly compositional systems, and hinted they can be described by means of categories with some additional feature. The natural question to ask now is if this is always the case.

We postpone this question, and instead we introduce the lack of compositionality as the other feature we aim to discuss.

EMERGENCE

We speak of **emergent effect** (or generative effect, e.g. in Fong, Spivak, *loc. cit.*) when the behaviour of a complex system cannot be reduced to the behaviours of its component.

Emergent effects occur pervasively in science, nature, linguistic, system theory etc.

Emergence: examples

- Life is an emergent phenomenon of chemistry.
- Mind is an emergent phenomenon of brain.
- Chemistry is an emergent phenomenon of physics.
- The stock market is an emergent phenomenon a system of companies, corporations, human organizations, media.
- Disease can be an emergent phenomenon of a creature biology.

OK, then..., with our compositional approach, it seems we are giving up explaining life, mind, diseases and many other fundamental matters.

In fact, this is not the case, but in order to explain, we need to dig a bit more in the categorical theoretical facets of the relation between compositionality and emergence.

Functorial semantics

A giant of (not only) Category Theory is indubitably William F. Lawvere.

- Lawvere's 1963's PhD thesis "Functorial Semantics of Algebraic Theories" deeply influences algebra, logic, computer science.

(A version of) **functorial semantics** is precisely what we need now to formalize the relationship between compositionality and emergence.

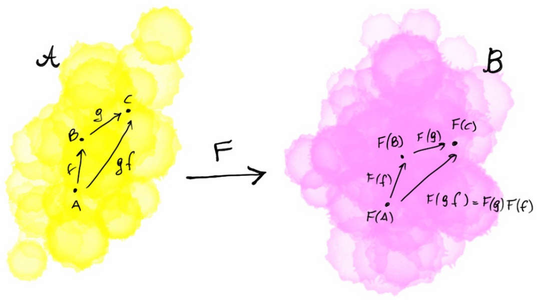
To start with, let's say what is a **functor**.

So far, we have considered just one given category, but it is full of categories out there!

Functors are able to connect different categories.

Functorial semantics

Suppose we have two categories $\mathcal{A}$ and $\mathcal{B}$. A functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is...



...well there is a technical definition of a **law** that assigns objects and morphisms of $\mathcal{A}$ to objects and morphisms of $\mathcal{B}$ in a way that respects some specified structure.

Functorial semantics

In Lawvere's thesis, $\mathcal{A}$ is an algebraic theory, $\mathcal{B}$ is a (mathematical) category and F realizes inside $\mathcal{B}$ the algebraic entities whose description is encoded in $\mathcal{A}$.

$$\mathcal{A} \xrightarrow{F} \mathcal{B}$$

Less formally, Lawvere says a functor is an **interpretation** of $\mathcal{A}$ within $\mathcal{B}$.

For example, one can think that the category $\mathcal{A}$ encodes the syntax, while the category $\mathcal{B}$ the semantic.

> "Questions of structure and constituency are settled by the *syntax* of L, while the meanings of simple expressions are given by the *lexical semantics* of L. Compositionality entails (although on many elaborations is not entailed by) the claim that syntax plus lexical semantics determines the entire semantics for L."
> *Compositionality, Stanford Encyclopedia of Philosophy.*

Emergence

In the study of complex systems, for a functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$:

- $\mathcal{A}$ encodes the compositional structure of the system,
- $\mathcal{B}$ is the category of the possible behaviours of the system.

Then, the functor F is an actual observation, and it describes a specified observed behaviour of the system.

One can have different categories of behaviours, and even if we fix one, one may have different observations of the same system, possibly during the same time span.

Emergence

An **emergent phenomenon** arises when the behaviour of a complex system cannot be explained by the behaviours of its simpler components.

> *"...while many real-world structures are compositional, the results of observing them are often not. The reason is that observation is inherently "lossy": in order to extract information from something, one must drop the details."* From: *B. Fong and D. Spivak, loc.cit., 2019.*

Loosing information is not necessarily a bad thing!

When we compute $3+1$, the result 4 loses the information that our 4 was made of a 3 and a 1... but this was precisely the purpose of the calculation!

Emergence

Here the system is Como Lake, and below there are three "observations": a photo picture, an old painting, a more recent painting.



There is a huge loss of information with respect to the real thing: an ecosystem 2.5 Km^3 of water surrounded by woods, mountains etc. But what remains is meaningful and expressive.

Emergence

Back to the question: **Do we live in a compositional world?**
Emerging phenomena would suggest a negative answer...

...indeed, they only suggest that we have non-compositional observations, while the underlying reality could still be of compositional nature.

More concretely, suppose we have a compositional description of a complex system, we can explain emergent phenomena as a **generative effect of our observations**.

Ontological remark: **emergence** is not a **feature of the system**, but of the **observables**.

However, because of this - and not despite of this - it deserves to be seriously investigated and quantified.

The functorial approach

The functorial approach can be a tool to deal with emergent effects.

A functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ **may preserve only some specified features of the (compositional) structure of the category** $\mathcal{A}$, while it may fail to preserve other features we are interested in.

However, it can provide a way to investigate and control such loss of information.
Generative effects arise where the compositional underlying structure of a system is not (all) preserved by the (functorial) observations.

This viewpoint is presented in *E. M. Adam. Systems, Generativity and Interactional Effects. PhD thesis, MIT, 2017*, where he develops a mathematical theory of interconnections to deal with cascade-like effects.

Is this the end?

Yes... but (maybe) it is also the beginning of new investigations and new interests.

To conclude, let me share the works and give credits to the people that inspired my talk today.

General

J. C. Baez, The mathematics of planet Earth. *Public Lecture at the 55th annual meeting of the South African Mathematical Society, 2012.*

Introductory texts

F. W. Lawvere, S. H. Schanuel, Conceptual Mathematics: A First Introduction to Categories. *Cambridge University Press, 1991.*

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Other references

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P. Schultz, D. I. Spivak, Temporal Type Theory: A topos-theoretic approach to systems and behavior. *Springer, Birkhäuser, 2019.*

J. C. Willems, The Behavioral Approach to Open and Interconnected Systems. *IEEE Control Systems Magazine 27 (6), 2007.*